Recall that the time derivative $\frac{\partial}{\partial t} g(t)$ is defined as

$$\left(\frac{\partial}{\partial t} g\right)(X, Y) = \underbrace{\frac{\partial}{\partial t} g(X, Y)}_{}$$

time-derivative of the smooth function $g(X, Y)$.

In local coordinates,

$$g(t) = g_{ij}(t)\, dx^i \otimes dx^j$$

$$\Rightarrow \quad \partial_t g(t) = \dot{g}_{ij}(t)\, dx^i \otimes dx^j.$$

$\therefore$ the time derivative of the metric is the time derivative of its components functions w.r.t. a fixed basis.

Similarly $\left(\frac{\partial}{\partial t} \nabla\right)(X, Y) = \frac{\partial}{\partial t} \nabla_X Y.$

now ∇ is NOT tensorial, but $\frac{\partial}{\partial t} \nabla$ is a tensor as

$$(\partial_t \nabla)(X, fY) = \frac{\partial}{\partial t}(\nabla_X fY) = \frac{\partial}{\partial t}(X(f)Y + f\nabla_X Y)$$

$$= f\,\partial_t \nabla_X Y = f\left(\frac{\partial}{\partial t}\nabla\right)(X, Y) \qquad \boxed{\text{\\\\\\}}$$

Variational Formulas

Given any smooth family of metrics it is desirable to compute the variations of all the associated quantities. We summarize them below for the case when

$$\partial_t g_{ij} = h_{ij}, \qquad h \in \Gamma(T^*M \otimes_s T^*M).$$

Lemma:-

$$\left(g^{ij} g_{jk} = \delta^i_k \Rightarrow (\partial_t g^{ij}) g_{jk} = -g^{ij} h_{jk}\right)$$

- $$\partial_t g^{ij} = -g^{ik} g^{jl} h_{kl}$$

proof on later pages.

- $$\partial_t \Gamma^k_{ij} = \frac{1}{2} g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})$$

- $$\partial_t R_{ijk}{}^{l} = \frac{1}{2} g^{lp} \left\{ \begin{aligned} &\nabla_i \nabla_j h_{kp} + \nabla_i \nabla_k h_{jp} \\ &- \nabla_i \nabla_p h_{jk} - \nabla_j \nabla_i h_{kp} \\ &- \nabla_j \nabla_k h_{ip} + \nabla_j \nabla_p h_{ik} \end{aligned} \right\}$$

proof on later pages.

$$= \frac{1}{2} g^{\ell p} \left\{ \begin{array}{l} \nabla_i \nabla_k h_{jp} + \nabla_j \nabla_p h_{ik} - \nabla_i \nabla_p h_{jk} \\ - \nabla_j \nabla_k h_{ip} - R^{q}_{ijk} h_{qp} - R^{q}_{ijp} h_{kq} \end{array} \right\}$$

- $$\partial_t R_{jk} = \frac{1}{2} g^{pq} \left(\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp} \right)$$

- $$\partial_t R = -\Delta(\mathrm{tr}\, h) + \nabla^p \nabla^q h_{pq} - \langle h, Rc \rangle$$

- $$\partial_t \, vol_g = \frac{\mathrm{tr}\, h}{2} \, vol$$ (proof below)

- $$\partial_t \int R \, vol_g = \int_M \left(\frac{R\, (\mathrm{tr}\, h)}{2} - \langle h, Rc \rangle \right) vol$$

Along the <u>RF</u> we have following improvements

$$\partial_t R = \Delta R + 2 |Rc|^2$$ — proof below.

$$\partial_t R_{jk} = \Delta R_{jk} + 2 g^{pq} g^{rs} R_{pjkr} R_{qs} - 2 g^{pq} R_{jp} R_{qk} .$$

$\downarrow$

proof :- $\partial_t R_{jk} = \Delta R_{jk} + \nabla_j \nabla_k R - g^{pq}(\nabla_q \nabla_j R_{kp} + \nabla_q \nabla_k R_{jp})$

$= \Delta R_{jk} + \nabla_j \nabla_k R - g^{pq} (\nabla_j \nabla_q R_{kp} - R_{qjkm} R_{mp} - R_{qjpm} R_{km} + \nabla_k \nabla_q R_{jp} - R_{qkjm} R_{mp} - R_{qkpm} R_{jm})$

$= \Delta R_{jk} + \nabla_j \nabla_k R - (\frac{1}{2} \nabla_j \nabla_k R + \frac{1}{2} \nabla_k \nabla_j R - R_{pjkm} R_{pm} + R_{jm} R_{km} - R_{pkjm} R_{pm} + R_{km} R_{jm})$

$=$ RHS.

Proof for $\partial_t R$ for RF

we have $\partial_t R = - \Delta (tr(-2Rc)) + div(div(-2Rc)) - \langle -2Rc, Rc \rangle$

$= 2\Delta R - \Delta R + 2|Rc|^2$ (we use twice contracted 2nd Bianchi)

$= \Delta R + 2|Ric|^2$.

Proof for the evolution of vol.

First recall that in local coordinates, the volume form

$$\mathrm{vol}_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^n$$

↳ det of $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

Recall that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A \quad \text{for a square matrix } A$$

where $\operatorname{adj} A$ = adjugate matrix = transpose of the cofactor matrix

The partial derivative of $\det A$ w.r.t. (i,j)-th entry is

$$\frac{\partial}{\partial a_{ij}} \det(A) = (-1)^{i+j} \det A_{ij}$$

$$= (\operatorname{adj} A)_{ji} = \det A \, (A^{-1})_{ji}$$

So

$$\frac{\partial}{\partial t} \sqrt{\det g_{ij}} = \frac{1}{2\sqrt{\det g_{ij}}} \frac{\partial}{\partial t} \det g = \frac{1}{2\sqrt{\det g}} \frac{\partial \det g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial t}$$

$$= \frac{1}{2\sqrt{\det g}} \det g \, (g^{-1})_{ji} \, h_{ij}$$

$$= \frac{1}{2} \sqrt{\det g} \; g^{ij} h_{ij}$$

$$\therefore \; \partial_t \mathrm{vol} = \frac{(\mathrm{tr}\, h)}{2} \mathrm{vol} .$$

Jacobi's formula

$$\frac{d}{dt} \det(A(t)) =$$

$$(\det A(t)) \cdot \mathrm{tr}\left(A(t)^{-1} \cdot \frac{dA(t)}{dt} \right)$$

for us $A(t) = g(t)$

$$\Rightarrow \frac{d}{dt} \det(g(t)) =$$

$$\det(g(t)) \cdot \mathrm{tr}\left(g(t)^{-1} \cdot h(t) \right)$$

The proofs for the evolutions of Rm, Ric, R and Γ for general variations can be done using the local coordinate expressions of these quantities and noticing that they are all components of a tensor (Γ is not but $\partial_t \Gamma$ is) and hence we can simplify our calculations by working in normal coordinates at a point.

We did this in detail in the class and the proofs in these notes are given below.

——— x ——— x ———

$$\Gamma^{k}_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$

$\Rightarrow$

$$\partial_t \Gamma^{k}_{ij} = \frac{1}{2} \left(\partial_t g^{kl} \right) \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right)$$
$$+ \frac{1}{2} g^{kl} \left(\partial_i \partial_t g_{jl} + \partial_j \partial_t g_{il} - \partial_l \partial_t g_{ij} \right)$$

At $p \in M^n$, choose geodesic normal coordinates so that $\Gamma^{k}_{ij}(p) = 0 \Rightarrow \partial_i g_{jl}(p) = 0$.
$\forall i, j, l.$

Also, $\partial_i A_{jk} = \nabla_i A_{jk}$ for any tensor.

$\therefore$

$$\frac{\partial}{\partial t} \Gamma^{k}_{ij}(p) = \frac{1}{2} g^{kl} \left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij} \right)(p).$$

For the Riemann curvature tensor :-

$$\partial_t R^{l}_{ijk} = \partial_i \left(\partial_t \Gamma^{l}_{jk} \right) - \partial_j \left(\partial_t \Gamma^{l}_{ik} \right)$$
$$+ \partial_t \left(\Gamma^{p}_{jk} \right) \cdot \Gamma^{l}_{ip} + \Gamma^{p}_{jk} \cdot \partial_t \Gamma^{l}_{ip}$$
$$- \left(\partial_t \Gamma^{p}_{ik} \right) \cdot \Gamma^{l}_{jp} - \Gamma^{p}_{ik} \, \partial_t \Gamma^{l}_{jp}.$$

Again, geodesic normal coordinates at $p \in M^n$ gives

$$\partial_t R_{ijk}{}^{l}(p) = \nabla_i\left(\partial_t \Gamma_{jk}^{l}\right)(p) - \nabla_j\left(\partial_t \Gamma_{ik}^{l}\right)(p)$$

$$= \frac{1}{2} g^{lp}\Big(\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} - \nabla_i\nabla_p h_{jk}$$
$$- \nabla_j\nabla_i h_{kp} - \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\Big)$$

use Ricci identity

Let's look at the evolution of the Rm for the RF.

we have

$$\partial_t R_{ijk}{}^{l} = g^{lp}\Big(-\nabla_i\nabla_j R_{kp} - \nabla_i\nabla_k R_{jp} + \nabla_i\nabla_p R_{jk}$$
$$+ \nabla_j\nabla_i R_{kp} + \nabla_j\nabla_k R_{ip} - \nabla_j\nabla_p R_{ik}\Big)$$

$$= g^{lp}\Big(-\nabla_i\nabla_k R_{jp} + \nabla_i\nabla_p R_{jk} + \nabla_j\nabla_k R_{ip} - \nabla_j\nabla_p R_{ik}$$
$$+ R_{ijk}{}^{m} R_{mp} + R_{ijp}{}^{m} R_{km}\Big)$$

now

$$\Delta R_{ijk}{}^{l} = g^{pq}\nabla_p\nabla_q R_{ijk}{}^{l}$$

$$= g^{pq} \nabla_p \left(-\nabla_i R_{jqk}{}^{l} - \nabla_j R_{qik}{}^{l} \right)$$

$$= g^{pq} \left(-\nabla_p \nabla_i R_{jqk}{}^{l} - \nabla_p \nabla_j R_{qik}{}^{l} \right)$$

$$\begin{aligned} = g^{pq} \Big(& -\nabla_i \nabla_p R_{jqk}{}^{l} + R_{pij}{}^{r} R_{rqk}{}^{l} + R_{piq}{}^{r} R_{jrk}{}^{l} \\ & + R_{pik}{}^{r} R_{jqr}{}^{l} - R_{pir}{}^{l} R_{jqk}{}^{r} \\ & - \nabla_j \nabla_p R_{qik}{}^{l} + R_{pjq}{}^{r} R_{rik}{}^{l} + R_{pji}{}^{r} R_{qrk}{}^{l} \\ & + R_{pjk}{}^{r} R_{qir}{}^{l} - R_{pjr}{}^{l} R_{qik}{}^{r} \Big) \end{aligned}$$

note:- $g^{pq} \left(-\nabla_i \nabla_p R_{jqk}{}^{l} \right) = \nabla_i \left(g^{pq} \nabla_p R_{qjk}{}^{l} \right)$

$$= \nabla_i \left(\nabla^l R_{jk} - \nabla_k R_j{}^{l} \right)$$

$$g^{pq} \left(-\nabla_j \nabla_p R_{qik}{}^{l} \right) = -\nabla_j \left(\nabla^l R_{ik} - \nabla_k R_i{}^{l} \right)$$

$\therefore$

$$\Delta R_{ijk}{}^{l} = -\nabla_i \nabla_k R_j{}^{l} + \nabla_i \nabla^l R_{jk} + \nabla_j \nabla_k R_i{}^{l} - \nabla_j \nabla^l R_{ik}$$

$$+g^{pq}\Big(R_{pij}{}^{r}R_{rqk}{}^{l} + R_{piq}{}^{r}R_{jrk}{}^{l} + R_{pik}{}^{r}R_{jqr}{}^{l} - R_{pir}{}^{l}R_{jqk}{}^{r}$$

$$+ R_{pjq}{}^{r}R_{rik}{}^{l} + R_{pji}{}^{r}R_{qrk}{}^{l} + R_{pjk}{}^{r}R_{qir}{}^{l} - R_{pjr}{}^{l}R_{qik}{}^{r}\Big)$$

note $\quad R_{pij}{}^{r}R_{rqk}{}^{l} - R_{pji}{}^{r}R_{rqk}{}^{l}$

$$= -R_{ijp}{}^{r}R_{rqk}{}^{l}$$

as $R_{pij}{}^{r} + R_{ijp}{}^{r} + R_{jpi}{}^{r} = 0$

terms on contraction give

$$-R_{i}{}^{r}R_{jrk}{}^{l} - R_{j}{}^{r}R_{rik}{}^{l}$$

$\therefore$

$$\Delta R_{ijk}{}^{l} = -\nabla_i\nabla_k R_j{}^{l} + \nabla_i\nabla^{l}R_{jk} + \nabla_j\nabla_k R_i{}^{l} - \nabla_j\nabla^{l}R_{ik}$$

$$-R_i{}^{r}R_{jrk}{}^{l} + R_j{}^{r}R_{irk}{}^{l}$$

$$+g^{pq}\Big(-R_{ijp}{}^{r}R_{rqk}{}^{l} + R_{pik}{}^{r}R_{jqr}{}^{l} - R_{pir}{}^{l}R_{jqk}{}^{r} + R_{pjk}{}^{r}R_{qir}{}^{l} - R_{pjr}{}^{l}R_{qik}{}^{r}\Big)$$

putting this in $\partial_t R_{ijk}{}^l$ term gives. same term.

$$\boxed{\begin{aligned}\partial_t R_{ijk}{}^l = {} & \Delta R_{ijk}{}^l - R_i{}^r R_{rjk}{}^l - R_j{}^r R_{irk}{}^l \\ & - R_k{}^r R_{ijr}{}^l + R_r{}^l R_{ijk}{}^r \\ & + g^{pq}\big(R_{ijp}{}^r R_{rqk}{}^l - 2R_{pik}{}^r R_{jqr}{}^l \\ & + 2R_{pir}{}^l R_{jqk}{}^r \big).\end{aligned}}$$