Introduction to the Ricci Flow
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Problem Set 5
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Problems

(1) Consider the normalized Ricci flow on (M^2, g) and let f be the potential of the curvature. Define the quantity
$$H = R - r + |\nabla f|^2$$
where R is the scalar curvature and r is the average scalar curvature. Prove that along the NRF
$$\partial_t H = \Delta H - 2|M|^2 + rH$$
where $M = \nabla\nabla f - \frac{1}{2}\Delta f \cdot g$.

(2) Prove that on a solution of the normalized Ricci flow on a compact surface, there exists a constant C depending only on the initial metric such that
$$R - r \leq H \leq Ce^{rt}.$$

(3) Using the result about the lower bound on the scalar curvature along NRF proved in the lectures and Prob (2), prove that along the NRF $g(t)$ on a compact surface there exists a constant C depending only on the initial metric such that
- if $r < 0$, then $r - Ce^{rt} \leq R \leq r + Ce^{rt}$.
- if $r = 0$, then $\dfrac{-C}{1 + Ct} \leq R \leq C$.
- if $r > 0$, then $-Ce^{-rt} \leq R \leq r + Ce^{rt}$.

(4) Recall from PSet 3 that (M^n, g) is called a gradient Ricci soliton (GRS) if $R_{ij} + \nabla_i\nabla_j f = \lambda g_{ij}$ and it is called shrinking, steady and expanding if $\lambda > 0, \lambda = 0$ and $\lambda < 0$ respectively. We also proved in PSet 2 that on any gradient Ricci soliton we have $R + |\nabla f|^2 - 2\lambda f = C$ where C is a constant.
  (a) Prove that on a GRS, $\Delta f - |\nabla f|^2 = \lambda(n - 2f) - C$.
  (b) Prove that any compact steady GRS is Ricci flat (and hence Einstein). (**Hint:** Use the fact that $0 = -\int_M \Delta(e^{-f})dv = \int_M (\Delta f - |\nabla f|^2)e^{-f}dv$ to prove that f must be a constant.)
  (c) Using the same idea as in part (b), prove that for expanding GRS ($\lambda < 0$) f must again be a constant and hence any compact steady or expanding GRS is Einstein.