**Problem Set 4**
**Due date: 07.01.2024**

## Problems

(1) We proved that if $M$ is a closed manifold and $g(t)$, $t \in [0,T]$ and $\bar{g}(t)$, $t \in [0,\bar{T}]$ are two solutions of the Ricci flow and if $g(0) = \bar{g}(0)$, then $g(t) = \bar{g}(t)$ for all $t \in [0, \min\{T, \bar{T}\}]$. A solution $g(t), t \in [0,T)$ is called **maximal**, where either $T = \infty$ or if $T < \infty$ then there *does not* exist any solution to the Ricci flow $\hat{g}(t), t \in [0, T+\epsilon), \epsilon > 0$ with the initial condition $\hat{g}(0) = g(0)$.

Prove that given $(M^n, g_0)$ there exists a unique, maximal solution $g(t)_{t\in[0,T_{\max})}$ of the Ricci flow with initial value $g_0$. We call $T_{\max}$ the **maximal existence time** or **singular time**. (**Hint:** Use Zorn's Lemma.)

(2) Recall that the Riemann curvature tensor can be further decomposed as follows:

$$\mathrm{Rm} = \frac{R}{2n(n-1)}(g \odot g) + \frac{1}{n-2}(\mathring{\mathrm{Ric}} \odot g) + W \tag{0.1}$$

where $R$ is the scalar curvture, $\mathring{\mathrm{Ric}}$ is trace-free Ricci tensor, $\mathring{\mathrm{Ric}}_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, $W$ is the $Weyl$ tensor and $\odot$ is the **Kulkarni–Nomizu** product, which is a product of symmetric 2-tensors and is given for symmetric 2-tensors $P$ and $Q$ by

$$(P \odot Q)_{ijkl} = P_{il}Q_{jk} + P_{jk}Q_{il} - P_{ik}Q_{jl} - P_{jl}Q_{ik}.$$

(If you have not seen (0.1) before, look up Ricci decompostion or decomposition of Riemann curvature.)

**Fact:** The Weyl tensor vanishes in dimension 3.

Use (0.1) to prove that in dimension 3, the Ricci curvature of a solution to the Ricci flow evolves as

$$\partial_t R_{jk} = \Delta R_{jk} + 3RR_{jk} - 6R_{jp}R_{pk} + (2|\mathrm{Ric}|^2 - R^2)g_{jk}. \tag{0.2}$$

**Bonus** exercise which you should think about once we discuss the maximum principle for tensors is the following important result:

**Lemma.** Let $g(t)$ be a solution to the Ricci flow in dimension 3 with $g(0) = g_0$. If $g_0$ has positive (nonnegative) Ricci curvature then $g(t)$ has positive (nonnegative) Ricci curvature for as long as the solution exists.

(3) We say that a vector field $X$ is a **conformal vector field** if

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = \frac{2}{n}\mathrm{div}(X) g_{ij} \tag{0.3}$$

(if $\mathrm{div}(X) = 0$ then $X$ is Killing.)

(a) Prove that if $M^n, g$ is a closed Riemannian manifold with $\mathrm{Ric} < 0$ then there are no nonzero conformal vector fields. (**Hint:** Take divergence of (0.3) and use the same idea as in the proof of the Bochner formula.)

(b) Prove the following result due to Bourguignon and Ezin[1]. If $(M^n, g)$ is a closed manifold with $n \geq 3$ and if $X$ is a conformal vector field then

$$\int_M \langle \nabla R, X \rangle \,\mathrm{vol} = \int_M R \,\mathrm{div}X \,\mathrm{vol} = 0.$$

Here $R$ is the scalar curvature. This is a generalization of something called Kazdan–Warner identity which we'll use later in the course.

[1]"*Scalar Curvature Functions in a Conformal Class of Metrics and Conformal Transformations*", J-P. Bourguignon and J-P. Ezin, Transactions of the American Mathematical Society, 1987.