Problem Set 3
Due date: 10.12.2024

Problems

(1) Prove that the twice contracted Riemannian second Bianchi identity ($\operatorname{div} \operatorname{Ric} = \frac{1}{2}\nabla R$) is equivalent to the diffeomorphism invariance of the scalar curvature.

(2) Let M^n be a closed manifold. In this problem, we'll learn about special solutions to the Ricci flow called **self-similar solutions** which are also called **Ricci solitons**. We make few important definitions.

Let $g(t)$ be a Ricci flow on M^n on some $(a,b) \ni 0$ and let $g(0) = g_0$. We call $g(t)$ a **self-similar** solution to RF is there exist scalars $\lambda(t)$ and diffeomorphisms ϕ_t of M such that

$$g(t) = \lambda(t)\phi_t^*(g_0), \qquad \forall \;\; t \in (a,b). \tag{0.1}$$

Thus, $g(t)$ is a special solution to the Ricci flow equation which changes in time only by diffeomorphisms and scalings of the original metric g_0, hence the name self-similar. Since the Ricci flow is invariant under diffeomorphims, self-similar solutions can be seen as genealized fixed points of the Ricci flow when one view the solutions modulo diffeomorphism and scalings.

We say (M^n, g_0) is a **Ricci soliton** if

$$-2\operatorname{Ric}(g_0) = \mathcal{L}_X g_0 + 2\lambda g_0 \tag{0.2}$$

for some constant λ and some vector field X on M. Note that this is just a generalization of the concept of Einstein manifolds as if the vector field is Killing, i.e., if $\mathcal{L}_X g_0 = 0$ then we get and Einstein metric. We can rescale the metric to assume that $\lambda \in \{-1, 0, 1\}$. This gives rise to three types of Ricci solitons: **shrinking** if $\lambda = -1$, **steady** if $\lambda = 0$ and **expanding** if $\lambda = 1$. Also note that in coordinates (0.2) reads as $-2R_{ij} = \nabla_i X_j + \nabla_j X_i + 2\lambda g_{ij}$. In the special case when $X = \nabla f$, i.e., X is gradient of a function, we get $-R_{ij} = \nabla_i \nabla_j f + \lambda g_{ij}$ and in this case (0.2) is called a **gradient Ricci soliton**.

Question (a) Show that if g_{Euc} is the Euclidean metric on $\mathbb{R}^n$ then it can be viewed as a steady Ricci soliton (so find X which makes that happen) and also an expanding gradient Ricci soliton for $\lambda = 1$ and $f(x) = \frac{1}{2}|x|^2$. The latter formulation is called the **Gaussian soliton.**

Question (b) We prove here that there is a bijection between the self-similar solutions of RF and Ricci solitons. First prove that if we have a self-similar solution $g(t)$ (i.e., it satisfies (0.1)) then indeed you can find λ and X such that (0.2) is satisfied for g_0.

Next, prove that if g_0 satisfies (0.2) then you can find a scalings $\lambda(t)$ and diffeomorphism ϕ_t such that $g(t)$ of the form in (0.1) is a solution to RF. In fact, show that $\lambda(t) = 1 + 2\lambda t$ and diffeomorphisms generated by the vector field $Y_t(x) = \dfrac{X(x)}{\lambda(t)}$ give you self-similar solutions.

Remark: Thus, we can talk about self-similar solutions and Ricci solitons interchangeably. Ricci solitons are very useful in understanding the "singularities" of the Ricci flow, i.e., points on M where the norm of the Riemann tensor will blow-up. If you remember the heuristic discussed in the class about "magnifying" at such a singularity then mathematically you are just doing scalings and pulling-back by diffeomorphisms and thus get a self-similar solution. This problem then allows us to use the very concrete (0.2) to study more about the properties of such solutions.

Just like the Gaussian soliton, there are other explicit examples of solitons. One such soliton is **Hamilton's cigar soliton** on $\mathbb{R}^2$ with another metric. It's an example of a steady soliton and hence it corresponds to a self-similar solution which turns out to be an **eternal solution**, i.e., it exists for all time.

(3) Prove the following identities for a gradient Ricci soliton.
  (a) $R + \Delta f = n\lambda$
  (b) $\nabla_i R = 2R_{im}\nabla^m f$
  (c) $R + |\nabla f|^2 - 2\lambda f =$ constant. (*Hint:* Look at $\nabla(R + |\nabla f|^2 - 2\lambda f)$ and use equations for gradient Ricci solitons to prove that the former expression is 0.)

In particular, part (c) has many applications in the study of Ricci solitons.