

Problem Set 1
Due date: 05.11.2024

Problems

- (1) Let $\{x^i\}$ and $\{y^\alpha\}$ be coordinate functions on a common open set. Using the fact that $g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$, show that

$$\Gamma_{\alpha\beta}^\gamma \frac{\partial x^k}{\partial y^\gamma} = \Gamma_{ij}^k \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta} + \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}.$$

- (2) Find the expression of the (3,1) Riemann curvature tensor in coordinates in terms of the Christoffel symbols. We will use the expression when we will calculate the variation formulas under the Ricci flow. (**Hint:** Proceed in the same as we obtained a formula for the Christoffel symbols in terms of the derivatives of the components of the metric.)
- (3) This question looks a bit longer because here we introduce the **Ricci curvature** and the **Scalar curvature** and prove some properties about them. Recall the following from linear algebra: If (V, \langle, \rangle) is an inner product space with basis $\{e_1, \dots, e_n\}$ and $A : V \rightarrow V$ is a linear map then $Ae_i = A_i^j e_j$ and $\text{tr}(A) = A_i^i \in \mathbb{R}$. Note that

$$g^{ij} \langle Ae_i, e_j \rangle = g^{ij} \langle A_i^l e_l, e_j \rangle = g^{ij} A_i^l g_{lj} = A_i^i = \text{tr}(A)$$

and thus $\text{tr}(A) = g^{ij} \langle Ae_i, e_j \rangle$.

If B is a bilinear form then we define $\text{tr}(B) = g^{ij} B_{ij}$.

Let (M, g) be a Riemannian manifold, $p \in M$ and fix $X_p, Y_p \in T_p M$. Define $A_p : T_p M \rightarrow T_p M$ by $A_p(Z_p) = R(Z_p, X_p)Y_p$, where R is the Riemann curvature tensor. Then from the above discussion we see that if $\{e_1, \dots, e_n\}$ is any basis of $T_p M$ then

$$\begin{aligned} \text{tr}(A_p) &= g(A_p e_i, e_j) g^{ij} \\ &= g(R(e_i, X_p)Y_p, e_j) g^{ij} \\ &= R(e_i, X_p, Y_p, e_j) g^{ij}. \end{aligned}$$

We can now make the following definition

Definition The **Ricci tensor** of g is the (2,0)-tensor Ric defined by

$$\text{Ric}(X, Y) = g^{ij} R(e_i, X, Y, e_j). \quad (0.1)$$

In local coordinates,

$$\text{Ric} = R_{jk} dx^j \otimes dx^k \quad \text{with} \quad R_{jk} = R_{ijkl} g^{il} \quad (0.2)$$

and thus Ric is obtained from the (4,0)-Riemann tensor by tracing on the first and the last index.

Definition The **scalar curvature** R of g is the trace of the Ricci tensor and is a function on M . That is, $R = \text{tr}_g(\text{Ric}) = g^{ij} R_{ij}$.

Prove the following:

- Ric is a symmetric (2,0)-tensor.
 - $\nabla_i R_{ijmk} = \nabla_k R_{jm} - \nabla_m R_{jk}$.
 - $\nabla_i R_{ij} = \frac{1}{2} \nabla_j R$ which in invariant form reads as $\text{div Ric} = \frac{1}{2} dR$ where div is the divergence of a tensor.
- (4) Let g be a Riemannian metric on a manifold M . Let $c > 0$ be a constant. Then cg is again a Riemannian metric on M as is said to be a rescaled metric. Show that as a (3,1)-tensor $R_{(3,1)}(cg) = R_{(3,1)}(g)$. Find the scalings of the (4,0)-Riemann curvature tensor, the Ricci curvature and the Scalar curvature. (**Hint:** What can you say about the Levi-Civita connection ∇^{cg} in terms of $\nabla^{g?}$)