Problem Set 2

① a)

$$\Delta|\nabla f|^2 = \nabla^i\nabla_i(\nabla_j f\,\nabla^j f)$$

$$= \nabla^i(2(\nabla_i\nabla_j f)\nabla^j f)$$

$$= 2(\nabla_i\nabla_j f)\nabla^i\nabla^j f + 2\nabla^j f\,\nabla^i\nabla_i\nabla_j f$$

$$= 2|\nabla\nabla f|^2 + 2\nabla^j f\,\nabla^i\nabla_j\nabla_i f \quad (\text{using } \nabla_i\nabla_j f = \nabla_j\nabla_i f)$$

$$= 2|\nabla\nabla f|^2 + 2\nabla^j f\left(\nabla_j\nabla^i\nabla_i f - R^i{}_{jim}\nabla^m f\right)$$

$$= 2|\nabla\nabla f|^2 + 2\nabla^j f\,\nabla_j(\Delta f) + 2R_{jm}\nabla^j f\,\nabla^m f.$$

and we get the other conclusion as well.

b) We integrate the equation in part a) and use Stokes's theorem to get

$$\int \Delta|\nabla f|^2 \mathrm{vol} = 0 = 2\int|\nabla\nabla f|^2 + 2\int Ric(\nabla f,\nabla f)\,\mathrm{vol}$$
$$+ 2\int \nabla_i f\,\nabla^i(\Delta f)\,\mathrm{vol}$$

$$\Rightarrow 0 = \left(\int|\nabla\nabla f|^2 + \int Ric(\nabla f,\nabla f) - \int \nabla^i\nabla_i f\,(\Delta f)\right)\mathrm{vol}$$

(integration by parts)

$$\Rightarrow \int |\nabla\nabla f|^2 \, vol + \int Ric(\nabla f, \nabla f)\, vol = \int (\Delta f)^2 \, vol.$$

and we get the second inequality as well.

c) Suppose $\Delta f = -\lambda f$, $\lambda > 0$ and assume $Ric \geq (n-1)K$.

Then from eq. (0.4) we get

$$\int Ric(\nabla f, \nabla f)\, vol \geq \int (n-1)K |\nabla f|^2 \, vol$$

$\cap$

and $$\int \frac{n-1}{n} (\Delta f)^2 \, vol = \int \frac{n-1}{n} \lambda^2 f^2 \, vol$$

$\therefore$

$$(n-1)K \int |\nabla f|^2 \, vol \leq \left(\frac{n-1}{n}\right) \lambda^2 \int f^2 \, vol$$

$$\Rightarrow \quad nK \int |\nabla f|^2 \, vol \leq \lambda^2 \int f^2 \, vol$$

$$= \lambda \int \lambda f \cdot f \, vol$$

$$= \lambda \int -\Delta f \cdot f \, vol$$

$$= \lambda \int |\nabla f|^2 \, vol \quad \text{[Integration by parts]}.$$

$$\Rightarrow \quad \lambda \geq nK \quad \boxed{\text{///}}$$

2) Want to prove

$$\left\langle \frac{\partial}{\partial t}\nabla_X Y, Z\right\rangle = \frac{1}{2}\left[(\nabla_Y h)(X,Z) + (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y)\right]$$

when $\partial_t g = h$.

note that $\because$ $\partial_t \nabla_X$ is a tensor, we get

$$\partial_t\left(g(\nabla_X Y, Z)\right) = (\partial_t g)(\nabla_X Y, Z) + g(\partial_t \nabla_X Y, Z)$$

$\Rightarrow$

$$g\left(\partial/\partial t\, \nabla_X Y, Z\right) = \frac{\partial}{\partial t}\left(g(\nabla_X Y, Z)\right) - (\partial_t g)(\nabla_X Y, Z)$$

$$= \frac{\partial}{\partial t}\left(X(g(Y,Z)) - g(Y, \nabla_X Z)\right) - h(\nabla_X Y, Z)$$

$$= X(h(Y,Z)) - h(Y, \nabla_X Z) - g(Y, \partial_t \nabla_X Z) - h(\nabla_X Y, Z)$$

$$= (\nabla_X h)(Y,Z) \underbrace{- g\left(Y, \frac{\partial}{\partial t}\nabla_X Z\right)}_{= -g\left(Y, \frac{\partial}{\partial t}\nabla_Z X\right)}$$

we reiterate this procedure

$$= (\nabla_X h)(Y,Z) - g\left(\frac{\partial}{\partial t}\nabla_Z X, Y\right)$$

$$= (\nabla_X h)(Y,Z) - \left((\nabla_Z h)(X,Y) - g\left(\frac{\partial}{\partial t}\nabla_Y Z, X\right)\right)$$

$$= (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y) + g\left(\frac{\partial}{\partial t}\nabla_Y Z, X\right)$$

$$= (\nabla_X h)(Y,Z) - (\nabla_Z h)(X,Y) + \left((\nabla_Y h)(Z,X) - g\left(\frac{\partial}{\partial t}\nabla_X Y, Z\right)\right)$$

(reiterating once more).

and hence we get the result.

∎

3) We use the norm for $i \neq j$, $|e_i \wedge e_j|^2 = |e_i \otimes e_j - e_j \otimes e_i|^2 = 1^2 + 1^2 = 2$ on $\Lambda^2 T^* M^3$. We use the result from the lecture notes about the Ricci curvature being the average of sectional curvature, we get

$$Ric(e_1, e_1) = \frac{1}{2}\left(Rm(e_1 \wedge e_2, e_1 \wedge e_2) + Rm(e_1 \wedge e_3, e_1 \wedge e_3)\right)$$

$$= \frac{1}{2}(\lambda_3 + \lambda_2)$$

Similarly, $Ric(e_2, e_2) = \dfrac{\lambda_1 + \lambda_3}{2}$, $Ric(e_3, e_3) = \dfrac{\lambda_1 + \lambda_2}{2}$

now, note that

$$Ric(e_1, e_2) = \frac{Ric(e_1 + e_2, e_1 + e_2) - Ric(e_1, e_1) - Ric(e_2, e_2)}{2}$$

$\therefore$ if we show that $Ric(e_1 + e_2, e_1 + e_2) = Ric(e_1, e_1) + Ric(e_2, e_2)$ then we are done.

This can be seen from the fact that if $\{e_1, e_2, e_3\}$ is an o.n.b. of $T_p M^3$ then so is $\left\{\frac{e_1 + e_2}{\sqrt{2}}, \frac{e_1 - e_2}{\sqrt{2}}, e_3\right\}$ so we can do the same calculation again and get that

$$Ric(e_1 + e_2, e_1 + e_2) = Rc(e_1, e_1) + Rc(e_2 + e_2)$$

$$\Rightarrow \quad Rc = \frac{1}{2}\begin{bmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_1 + \lambda_3 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{bmatrix} \quad \text{and } R = \lambda_1 + \lambda_2 + \lambda_3.$$

b) if $Ric = \lambda g \Rightarrow R = \lambda n \Rightarrow \lambda = \frac{R}{n}$

$\therefore \quad Ric = \frac{R}{n} g$

differentiating both sides and using 2^{nd} Bianchi identity, we get

$$\frac{1}{2} \nabla R = \frac{1}{n} \nabla R \Rightarrow \nabla R = 0 \Rightarrow R \text{ constant.}$$

Also for $n = 3$, $Ric = \frac{1}{2} \begin{pmatrix} \lambda_1 + \lambda_2 & 0 & 0 \\ 0 & \lambda_3 + \lambda_1 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{pmatrix} = \frac{R}{3} g = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$

$\Rightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = \text{constant}.$

4) a) $\Leftarrow$ $\mathrm{Ric} = \frac{1}{n} \frac{\int R \, \mathrm{vol}}{\int \mathrm{vol}} g(t)$

M

$\Rightarrow \quad R = \frac{\int R \, \mathrm{vol}}{\int \mathrm{vol}}$ (by tracing)

$\Rightarrow dR = 0$ (by Stokes' theorem) $\Rightarrow R = \text{constant}$

$\therefore \quad \mathrm{Ric} = \frac{\text{constant}}{n} g \Rightarrow$ Einstein manifold.

Other direction is easy.

b)

RF and NRF differ only by a scaling of space and time.

Let $g(t)$ be a RF. and suppose M has finite volume.

define dilations $\lambda(t) > 0$ s.t. $\tilde{g}(t) = \lambda(t) g(t)$

satisfy $\int_{M^n} \widetilde{\mathrm{vol}}_{\tilde{g}(t)} = 1.$

Goal:- Show that

$$\frac{\partial \tilde{g}}{\partial \tilde{t}} = -2\widetilde{\mathrm{Ric}}(\tilde{g}(t)) + \frac{2}{n} \frac{\int_M \tilde{R} \, \widetilde{\mathrm{vol}}}{\int_M \widetilde{\mathrm{vol}}} \tilde{g}(t)$$

Let $\tilde{t} = \int_0^t \lambda(\tau) \, d\tau.$

then $\frac{d\tilde{t}}{dt} = \lambda(t).$

we exactly know how geometric quantities scale as per the scaling of the metrics.

Then : $\frac{d}{\partial t} \int vol_{g(t)} = -\int R \, vol$

$\therefore$

$$\frac{\partial}{\partial \tilde{t}} \tilde{g}(t) = \frac{dt}{d\tilde{t}} \cdot \frac{\partial}{\partial t}\left(\lambda(t) g(t)\right)$$

$$= \frac{1}{\lambda(t)} \cdot \left(-2Rc(t) \cdot \lambda(t) + \partial_t \lambda(t) g(t)\right)$$

$$= -2\overline{Rc}(t) + \frac{\partial_t \lambda(t)}{\lambda^2(t)} \bar{g}(t) \quad \text{— (1)}$$

Go back to the evolution eqn of the vol. form and notice

$$\partial_t \log \det(\lambda(t) g(t)) = -2R + \frac{n}{\lambda(t)} \frac{d\lambda(t)}{dt}$$

and : $\int \widetilde{vol} = 1$ (by assumption)

$$\Rightarrow \quad 0 = \frac{d}{dt} \int \widetilde{vol} = \int \frac{\partial}{\partial t} \log \sqrt{\det(\lambda(t) g(t))} \, \widetilde{vol}$$

$$= \int \left(-\lambda \tilde{R} + \frac{n}{2\lambda} \frac{\partial \lambda}{\partial t}\right) \widetilde{vol}$$

$$= -\lambda \frac{\int \tilde{R} \, \widetilde{vol}}{\int \widetilde{vol}} + \frac{n}{2\lambda} \frac{\partial \lambda}{\partial t} \underbrace{\int \widetilde{vol}}_{=1}$$

$\therefore \quad \dfrac{\lambda \int \tilde{R}\, \widetilde{vol}}{\int \widetilde{vol}} = \dfrac{n}{2\lambda} \dfrac{d\lambda}{dt}$

$$\Rightarrow \quad \frac{2}{n} \frac{\int \tilde{R}\, \widetilde{vol}\, \tilde{g}(t)}{\int \widetilde{vol}} = \frac{1}{\lambda^2(t)} \frac{d\lambda}{dt} \tilde{g}(t) \quad \text{—} \; 2$$

$\therefore$ from 1 and 2 we get

$$\boxed{\frac{\partial \tilde{g}(t)}{\partial \tilde{t}} = -2\widetilde{Ric} + \frac{2}{n} \frac{\int \tilde{R}\, \widetilde{vol}\, \tilde{g}(t)}{\int \widetilde{vol}}}$$

→ so starting from a solⁿ $g(t)$ of the RF, by dilating space and time we got a solⁿ of the normalized Ricci flow.