Hints for Pset 1

1. We know that

$$g_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^j}{\partial y^\beta}$$

(this can be seen by the fact that if $\tau : M \to N$ then

$$(\tau^* g)(X, Y) = g(\tau_* X, \tau_* Y) \text{ and in coordinates}$$

$$(\tau^* g)_{\alpha\beta} = (\tau^* g)\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta}\right).)$$

Let us use the notation $\varphi^i_\alpha = \dfrac{\partial x^i}{\partial y^\alpha}$ and hence $g_{\alpha\beta} = g_{ij} \varphi^i_\alpha \varphi^j_\beta$

and $g^{\gamma\delta} = g^{k\ell} (\varphi^{-1})^\gamma_k (\varphi^{-1})^\delta_\ell$. We now compute

$$\Gamma^\gamma_{\alpha\beta} \varphi^k_\gamma = \frac{1}{2} g^{k\ell} (\varphi^{-1})^\delta_\ell \left(\frac{\partial}{\partial y^\alpha} (g_{jm} \varphi^j_\beta \varphi^m_\delta) + \frac{\partial}{\partial y^\beta} (g_{im} \varphi^i_\alpha \varphi^m_\delta) - \frac{\partial}{\partial y^\delta} (g_{ij} \varphi^i_\alpha \varphi^j_\beta) \right)$$

now, $\varphi^i_\alpha \varphi^j_\beta \Gamma^k_{ij} = \varphi^i_\alpha \varphi^j_\beta \frac{1}{2} g^{kl} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$

and

$$\frac{1}{2} g^{kl} (\varphi^{-1})^\delta_\ell \left(\frac{\partial}{\partial y^\alpha} g_{jm}\right) \varphi^j_\beta \varphi^m_\delta = \frac{1}{2} g^{kl} \varphi^j_\beta (\varphi^{-1})^\delta_\ell \varphi^m_\delta \frac{\partial g_{jm}}{\partial y^\alpha}$$

$$= \frac{1}{2} g^{kl} \varphi^j_\beta \frac{\partial g_{j\ell}}{\partial x^i} \varphi^i_\alpha$$

and similarly for the other derivative of the metric components terms.

$\therefore$ we get

$$\Gamma^{\gamma}_{\alpha\beta}\varphi^{k}_{\gamma} = \varphi^{i}_{\alpha}\varphi^{j}_{\beta}\Gamma^{k}_{ij} + \frac{1}{2}g^{kl}\left[\underline{g_{jl}}\frac{\partial\varphi^{j}_{\beta}}{\partial y^{\alpha}} + \underline{g_{il}}\frac{\partial\varphi^{i}_{\alpha}}{\partial y^{\beta}} - \underline{g_{ij}\left((\varphi^{-1})^{\delta}_{l}\right.}\left.\frac{\partial\varphi^{i}_{\alpha}}{\partial y^{\delta}}\right)\varphi^{j}_{\beta}\right]$$

$$+ \underline{\frac{1}{2}g^{kl}(\varphi^{-1})^{\delta}_{l}\Big(g_{jm}\varphi^{j}_{\beta}}\frac{\partial\varphi^{m}_{\delta}}{\partial y^{\alpha}} + g_{im}\varphi^{i}_{\alpha}\frac{\partial\varphi^{m}_{\delta}}{\partial y^{\beta}} - g_{ij}\varphi^{i}_{\alpha}\frac{\partial\varphi^{j}_{\beta}}{\partial y^{\delta}}\Big)$$

$$= \varphi^{i}_{\alpha}\varphi^{j}_{\beta}\Gamma^{k}_{ij} + \frac{\partial^2 x^k}{\partial y^{\alpha}\partial y^{\beta}} \Rightarrow \Gamma \text{ is NOT a tensor.}$$

note $$\frac{\partial\varphi^{k}_{\beta}}{\partial y^{\alpha}} = \frac{\partial}{\partial y^{\beta}}\varphi^{k}_{\alpha} = \frac{\partial}{\partial y^{\alpha}}\left(\frac{\partial x^k}{\partial y^{\beta}}\right) = \frac{\partial^2 x^k}{\partial y^{\alpha}\partial y^{\beta}}$$

$$\frac{\partial\varphi^{i}_{\delta}}{\partial y^{\alpha}} = \frac{\partial^2 x^i}{\partial y^{\alpha}\partial y^{\delta}} = \frac{\partial^2 x^i}{\partial y^{\delta}\partial y^{\alpha}} = \frac{\partial}{\partial y^{\delta}}\varphi^{i}_{\alpha}$$

and $$\frac{\partial\varphi^{j}_{\delta}}{\partial y^{\beta}} = \frac{\partial}{\partial y^{\delta}}\varphi^{j}_{\beta}$$

Also, $-\frac{1}{2}g^{kl}g_{ij}(\varphi^{-1})^{\delta}_{l}\frac{\partial\varphi^{i}_{\alpha}}{\partial y^{\delta}}\varphi^{j}_{\beta}$

and $\frac{1}{2}g^{kl}g_{jm}(\varphi^{-1})^{\delta}_{l}(\varphi^{j}_{\beta})\frac{\partial\varphi^{m}_{\delta}}{\partial y^{\alpha}}$ same

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2.) The (3,1) Rm tensor is

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Let $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$, to get

$$R_{ijk}{}^{l} \partial_l = R(\partial_i, \partial_j)\partial_k = \nabla_i \nabla_j \partial_k - \nabla_j \nabla_i \partial_k - 0$$

$$= \nabla_i \left(\Gamma_{jk}^{l} \partial_l\right) - \nabla_j \left(\Gamma_{ik}^{l} \partial_l\right) = \left(\nabla_i \Gamma_{jk}^{l}\right)\partial_l + \Gamma_{jk}^{l} \nabla_i \partial_l - \left(\nabla_j \Gamma_{ik}^{l}\right)\partial_l - \Gamma_{ik}^{l} \nabla_j \partial_l$$

$$= \frac{\partial \Gamma_{jk}^{l}}{\partial x^i} \partial_l + \Gamma_{jk}^{l} \Gamma_{il}^{m} \partial_m - \frac{\partial \Gamma_{ik}^{l}}{\partial x^j} \partial_l - \Gamma_{ik}^{l} \Gamma_{jl}^{m} \partial_m$$

$$= \left(\frac{\partial \Gamma_{jk}^{l}}{\partial x^i} - \frac{\partial \Gamma_{ik}^{l}}{\partial x^j} + \Gamma_{jk}^{m} \Gamma_{im}^{l} - \Gamma_{ik}^{m} \Gamma_{jm}^{l} \right) \partial_l .$$

$\boxtimes$

3)

a) $$R_{jk} = g^{il}\left(R_{ijkl}\right)$$
$$= g^{il}\left(-R_{jkil} - R_{kijl}\right)$$
$$= 0 - g^{il}R_{kijl} = +g^{il}R_{ikjl} = R_{kj}.$$

b) $$\nabla_i R_{jkml} + \nabla_m R_{jkli} + \nabla_l R_{jkim} = 0$$

$$\Rightarrow \quad g^{ij}\left(\ \cdot\ \cdot\ \cdot\ \right) = 0$$

$$\Rightarrow \quad \nabla_i R^i{}_{kml} + \nabla_m R_{kl} - \nabla_l R_{km} = 0$$

$$\Rightarrow \quad \nabla_i R^i{}_{kml} = \nabla_l R_{km} - \nabla_m R_{kl}.$$

further, $$g^{kl}\nabla_i R^i{}_{kml} = g^{kl}\nabla_l R_{km} - g^{kl}\nabla_m R_{kl}$$

$$\Rightarrow \quad -\nabla_i R^i{}_m = \nabla_l R^l{}_m - \nabla_m R$$

$$\Rightarrow \quad \frac{1}{2}\nabla_m R = \nabla_l R^l{}_m.$$

4). If $\tilde{g} = cg$ then $\tilde{g}^{-1} = c^{-1}g$

$$\Rightarrow \quad \tilde{\Gamma} = \Gamma \Rightarrow \tilde{R}^{3,1} = R^{(3,1)}$$

$$\tilde{R}^{(4,0)} = cR^{(4,0)}$$

$$\widetilde{Ric} = Ric$$

$$\tilde{R} = c^{-1}R.$$