Differential operators and basics of the Ricci flow

We recall that the square of the covariant derivative is

$$\nabla^2 : \Gamma(T^p_q M) \longrightarrow \Gamma(T^{p+2}_q M)$$

w/

$$\nabla^2_\alpha (X, Y, Z_1, \ldots, Z_p) = \nabla_X (\nabla\alpha)(Y, Z_1, \ldots, Z_p)$$

$$= [\nabla_X (\nabla\alpha(Y)) - \nabla\alpha(\nabla_X Y)](Z_1, \ldots, Z_p)$$

$$= \nabla_X (\nabla_Y \alpha)(Z_1, \ldots, Z_p) - \nabla_{\nabla_X Y}\alpha(Z_1, \ldots, Z_p)$$

so in short

$$\nabla^2_{X,Y}\alpha = \nabla_X \nabla_Y \alpha - \nabla_{\nabla_X Y}\alpha.$$

For ex. in local coordinates

$$\nabla_i R_{jk} = (\nabla Rc)(\partial_i, \partial_j, \partial_k)$$

$$= (\nabla_{\partial_i} Rc)(\partial_j, \partial_k)$$

$$= \frac{\partial}{\partial x^i} R_{jk} - \Gamma_{ij}^{l} R_{lk} - \Gamma_{ik}^{l} R_{jl}$$

Similarly

$$\nabla_i R_{jklm} = \partial_i R_{jklm} - \Gamma_{ij}^{p} R_{pklm} - \Gamma_{ik}^{p} R_{jplm} - \Gamma_{il}^{p} R_{jkpm} - \Gamma_{im}^{p} R_{jklp}$$

for a function f, one can show that

$$\nabla_i \nabla_j f = (\nabla \nabla f)(\partial_i, \partial_j)$$

$$= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^{k} \frac{\partial f}{\partial x^k} .$$

* We denote the m-th order derivative of α by $\nabla^m \alpha = \underbrace{\nabla \cdots \nabla}_{m\text{-times}} \alpha$

and its components will be denoted by

$$\nabla_{j_1} \cdots \nabla_{j_m} \alpha_{i_1 \ldots i_p} := (\nabla^m \alpha)\left(\frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_m}}, \frac{\partial}{\partial x^{i_1}} \cdots \partial/\partial x^{i_r}\right).$$

How to understand the iterated derivatives?

Suppose T is a tensor and we want to calculate

$$\nabla\nabla T(X,Y,\omega) = (\nabla_X(\nabla T))(Y,\omega)$$

$$= \nabla_X\big((\nabla T)(Y,\omega)\big) - (\nabla T)(\nabla_X Y,\omega) - (\nabla T)(Y,\nabla_X \omega)$$

$$= \nabla_X\Big(\big(\underbrace{\nabla_Y T}_{\text{a v.f. applied on } \omega}\big)(\omega)\Big) - (\nabla_{\nabla_X Y} T)(\omega) - (\nabla_Y T)(\nabla_X \omega)$$

$$= X\big((\nabla_Y T)(\omega)\big) - (\nabla_{\nabla_X Y} T)(\omega) - (\nabla_Y T)(\nabla_X \omega)$$

$$= (\nabla_X \nabla_Y T)(\omega) + \cancel{(\nabla_Y T)(\nabla_X \omega)} - (\nabla_{\nabla_Y X} T)(\omega) - \cancel{(\nabla_Y T)(\nabla_X \omega)}$$

$$= (\nabla_X \nabla_Y T)(\omega) - (\nabla_{\nabla_Y X} T)(\omega)$$

and hence the result.

Same thing happens for higher degree tensors as well.

Lie Derivative

For $X, Y \in \Gamma(TM)$

$\mathcal{L}_X Y = [X, Y]$ w/

$[X, Y] \in \Gamma(TM)$ s.t.

$[X, Y](f) = X(Y(f)) - Y(X(f))$.

If θ is a 1-form then

$(\mathcal{L}_X \theta)(Y) = X(\theta(Y)) - \theta([X, Y])$

even though one doesn't need a metric to define the Lie derivative, it is related to the L-C connection via the formula

$(\mathcal{L}_X A)(Y_1, \dots Y_p, \theta_1, \dots, \theta_q)$

$= X(A(Y_1, \dots, Y_p, \theta_1, \dots, \theta_q))$

$- \sum_{1 \le i \le p} A(Y_1, \dots, [X, Y_i], \dots, Y_p, \theta_1, \dots, \theta_q)$

$- \sum_{1 \le j \le q} A(Y_1, \dots, Y_p, \theta_1, \dots, \mathcal{L}_X \theta_j, \dots, \theta_q)$

In particular, when $A = g$ then

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$$

whose expression in coordinates give

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i .$$

If $X = \nabla f$ for $f \in C^\infty(M)$ then

$$(\mathcal{L}_X g)_{ij} = (\mathcal{L}_{\nabla f} g)_{ij} = \nabla_i \nabla_j f + \nabla_j \nabla_i f$$

$$= 2 \nabla_i \nabla_j f .$$

for v.f. $\nabla^2 Z(X, Y) - \nabla^2 Z(Y, X) = R(X, Y) Z$

Ricci Identities

$= 0$ $\quad \nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k = R_{ijm}{}^k Z^k$

We have

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_{k_1, \dots, k_r} = -\sum_{l=1}^{r} R_{ijk_l}{}^m \alpha_{k_1 \cdots k_{l-1} m k_{l+1} \cdots k_r}$$

So for a 1-form α,

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_k = - R_{ijkl} \alpha_l$$

or for a 2-tensor β

$$\nabla_i \nabla_j \beta_{kl} - \nabla_j \nabla_i \beta_{kl} = - R_{ijkm} \beta_{ml} - R_{ijlm} \beta_{km} .$$

The divergence of a $(p,0)$-tensor is

$$(\operatorname{div}(\alpha))_{i_1 \dots i_{p-1}} = g^{jk} \nabla_j \alpha_{k i_1 \dots i_{p-1}}$$

$$= \nabla_j \alpha^j{}_{i_1 \dots i_{p-1}}$$

for 1-forms α, $\operatorname{div} \alpha$ is a function,

$$\operatorname{div} \alpha = \nabla_i \alpha^i .$$

Laplacian

Laplacian Δ on functions is div (grad)

i.e,

$$\Delta = \operatorname{div} \nabla = g^{ij} \nabla_i \nabla_j$$

$$= g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right)$$

For tensors, again

$$\Delta = \operatorname{div}(\operatorname{grad}) = \operatorname{trace}_g \nabla^2$$

$$= g^{ij} \nabla_i \nabla_j$$

<u>Exercise</u>:- Prove the Bochner formula for $|\nabla f|^2$

i.e, $\forall f \in C^\infty(M)$

$$\Delta |\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla_i f \nabla_j f + 2\nabla_i f \nabla_i(\Delta f).$$

Conclude that if $Rc \geq 0$, $\Delta f \equiv 0$ and $|\nabla f| = 1$ then ∇f is parallel.

We also have the divergence theorem and integration by parts formula. Let M^n be a closed manifold and $u, v \in C^\infty(M)$ and $X \in \Gamma(TM)$. Then

$$\int_M \operatorname{div} X \, vol = 0 \quad \text{and so}$$

$$\int_M \Delta u \, vol = 0$$

$$\int_M u \Delta v \, vol = \int_M v \Delta u \, vol \quad \text{(Integration by parts).}$$

note:- $0 = \int \operatorname{div}(v \operatorname{grad} u) = \int \nabla^i (v \nabla_i u) vol$

$$= \int \langle \nabla v, \nabla u \rangle vol + \int v \Delta u \, vol$$

$$\Rightarrow \int v \Delta u = -\int \langle \nabla v, \nabla u \rangle vol$$

$$= \int u \Delta u \, vol. \quad \text{and so on.}$$