Symmetries of R

$R(X,Y,Z,W) := g(R(X,Y)Z, W)$

↓

(4,0) tensor obtained from (3,1) R by musical isomorphisms.

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}^{\ \ \ l}\partial_l$$

$$R(\partial_i, \partial_j, \partial_k, \partial_m) = R_{ijkm}$$

$$\boxed{R_{ijkm} = R_{ijk}^{\ \ \ l} g_{lm}}$$

Prop :-

a) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

b) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

c) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W)$

$= 0$

d) $R(X, Y, Z, W) = R(Z, W, X, Y)$.

a) is always true and w/ b) allows us to see $R \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$ i.e, as a symmetric bilinear forms on the space of 2-forms.

b) follows from metric compatibility, $\nabla g = 0$

c) is true for any torsion free connection on TM. It is called the First Bianchi identity.

d) follows from a), b) and c).

In local coordinates, these read as

1) $R_{ijkl} = -R_{jikl}$

2) $R_{ijkl} = -R_{ijlk}$

3) $R_{ijkl} + R_{jkil} + R_{kijl} = 0$

4) $R_{ijkl} = R_{klij}$

Proof :- a) done

b) since $\nabla g = 0 \Rightarrow$

$$Y(g(Z,Z)) = 2\, g(\nabla_Y Z, Z)$$

$$X(Y(g(Z,Z))) = 2\, X(g(\nabla_Y Z, Z))$$
$$= 2 g(\nabla_X \nabla_Y Z, Z) \longrightarrow (1)$$

Exercise :- Prove c) using local coordinates.

$$+ 2g(\nabla_y Z, \nabla_x Z)$$

$$Y(X(g(Z,Z))) = 2g(\nabla_y Z, \nabla_x Z) + 2g(Z, \nabla_y \nabla_x Z) \quad -(2)$$

$$[X,Y](g(Z,Z)) = 2g(Z, \nabla_{[X,Y]} Z) \quad -(3)$$

(1) − (2) − (3)

$$X(Y(g(Z,Z))) - Y(X(g(Z,Z))) - [X,Y](g(Z,Z))$$

(b/c, we have $XY|Z|^2 - YX|Z|^2 - [X,Y]|Z|^2 = (XY - YX - [X,Y])|Z|^2$ $= 0$)

$$= 2R(X,Y,Z,Z) = 0$$

$\Rightarrow$ polarize to get (b).

$R(X,Y, Z+W, Z+W) = 0$

$\Rightarrow R(X,Y,Z,Z) + R(X,Y,Z,W) + R(X,Y,W,Z) + R(X,Y,W,W) = 0$

c) Want to show that

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

expand and use torsion-free. and use Jacobi identity for $[\cdot,\cdot]$.

Proof :- we prove that $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$.

This expression is

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y$$
$$- \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y$$

we manipulate this as

$$\nabla_X (\nabla_Y Z - \nabla_Z Y) + \nabla_Y (\nabla_Z X - \nabla_X Z) + \nabla_Z (\nabla_X Y - \nabla_Y X)$$
$$- \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y$$

By torsion-freeness of ∇,

$$\nabla_X ([Y,Z]) + \nabla_Y [Z,X] + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X$$
$$- \nabla_{[Z,X]} Y$$

now, notice that $[X,[Y,Z]] = \nabla_X [Y,Z] - \nabla_{[Y,Z]} X$

and so on

$\Rightarrow$ we get

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

By the Jacobi identity.

Jacobi identity for $[\cdot,\cdot]$.

d) Write identity c) in 4 ways.

Sectional Curvature

Let (M,g) be Riemann.

Given $X_p, Y_p \in T_pM$

$$|X_p \wedge Y_p|^2_{g_p} = |X_p|^2_{g_p} |Y_p|^2_{g_p} - g_p(X_p, Y_p)^2$$

$$|X \wedge Y|^2 = |X|^2 |Y|^2 - \langle X, Y\rangle^2$$

Defⁿ :- Let L_p be a 2-dimensional subspace of T_pM $(n \geq 2)$. Define the sectional curvature $K_p(L_p)$ of (M,g) at p in

"L_p direction" by

$$K_p(L_p) = \frac{R(X_p, Y_p, Y_p, X_p)}{|X_p \wedge Y_p|^2}$$

for any basis X_p, Y_p of L_p.

(denom. not zero as X_p, Y_p are basis).

if $\tilde{X} = aX + bY$

$\tilde{Y} = cX + dY$

$\tilde{X} \wedge \tilde{Y} = (ad - bc)\, X \wedge Y$

show that

$$\frac{R(\tilde{X}, \tilde{Y}, \tilde{Y}, \tilde{X})}{|\tilde{X} \wedge \tilde{Y}|^2} = \frac{R(X, Y, Y, X)}{|X \wedge Y|^2}$$

if $n = 2$, $L_p = T_pM \quad \forall p \in M$

$\Rightarrow$ sectional curvature is just a smooth function on M.

Lemma:- The sectional curvature determines the Riemann curvature and vice-versa.

Precisely, suppose V^n $(n \geq 2)$ is a $\mathbb{R}$-inner product space and R and $\tilde{R}$ be two trilinear maps s.t.

$\langle R(X,Y,Z), W \rangle$ and $\langle \tilde{R}(X,Y,Z), W \rangle$ are skew in X, Y, skew in Y, Z and satisfy 1st Bianchi identity.

Let $X, Y \in V$ be linearly independent.

Let $\sigma = \text{span}\{X, Y\}$

Define
$$K(\sigma) = \frac{\langle R(X,Y,Y), X \rangle}{|X \wedge Y|^2}$$

$$\tilde{K}(\sigma) = \langle \tilde{R}(X,Y,Y), X \rangle$$

$$\overline{\qquad\qquad\qquad}$$
$$|x \wedge y|^2$$

if $K = \tilde{K}$ $\forall$ $\sigma \subseteq V$ then $R = \tilde{R}$.

<u>Lemma</u> Let V be a real vector space w/ $\dim V \geq 2$ and R and $\tilde{R}$ be trilinear maps $V \times V \times V \to V$ satisfying

$$\langle R(x,y,z), w \rangle = (x,y,z,w)$$
$$\langle \tilde{R}(x,y,z), w \rangle = (x,y,z,w)^{\sim}$$

have the following symmetries :-

$$(x,y,z,w) = -(y,x,z,w) = -(x,y,w,z)$$
$$= (z,w,x,y)$$

and $(x,y,z,w) + (y,z,x,w) + (z,x,y,w) = 0$

same for $\sim$.

Let x, y be linearly independent. Let $\sigma = \text{span}\{x,y\}$

define $$K(\sigma) = \frac{(x,y,y,x)}{|x \wedge y|^2}$$

$$\tilde{K}(\sigma) = \frac{(x,y,y,x)^{\sim}}{|x \wedge y|^2}$$

If $\tilde{K}(\sigma) = K(\sigma)$ $\forall$ 2-dimensional subspace $\sigma \subseteq V$ then $R = \tilde{R}$.

Proof By hypo. $(x,y,y,x) = (x,y,y,x)^{\sim}$ $\forall x,y$.

polarize $(x+y, z, z, x+y) = (x+y, z, z, x+y)^{\sim}$

$\Rightarrow (x,z,z,y) + (y,z,z,x) = (x,z,z,y)^{\sim} + (y,z,z,x)^{\sim}$

$\Rightarrow 2(x,z,z,y) = 2(x,z,z,y)^{\sim}$

By symmetries

$\Rightarrow (x,z,z,y) = (x,z,z,y)^{\sim}$

polarize again, $z \longmapsto z+w$

$(x,z,w,y) + (x,w,z,y) = (x,z,w,y)^{\sim} + (x,w,z,y)^{\sim}$

$\Rightarrow (x, z, w, y) - (x, z, w, y)^{\sim} =$

$- (x, w, z, y) + (x, w, z, y)^{\sim}$

$= (w, x, z, y) - (w, x, z, y)^{\sim}$

$\Rightarrow$ $\sim$ is invariant under cyclic permutation

$x \mapsto z \mapsto w \mapsto x$

$$\Rightarrow \sum_{\substack{x, y, z \\ \text{cyclic}}} (x, z, w, y) - (x, z, w, y)^{\sim} = 3(x, z, w, y) - 3(x, z, w, y)^{\sim}$$

$\Rightarrow 0 - 0 = 0$

$\Rightarrow (x, z, w, y) = (x, z, w, y)^{\sim}$

2nd Bianchi Identity

Let (M, g) be Riemannian and R be the

what does $\nabla_i R_{jklm}$ mean?

$$\nabla_i R_{jklm} = \left(\nabla_{\frac{\partial}{\partial x^i}} Rm\right)(\partial_j, \partial_k, \partial_l, \partial_m) = \frac{\partial}{\partial x^i} R_{jklm} - \Gamma_{ij}^{p} R_{pklm} - \Gamma_{ik}^{p} R_{jplm}$$

$- \Gamma_{il}^{p} R_{jkpm} - \Gamma_{im}^{p} R_{jklp}$.

Riemannian (4,0) tensor. Then

$$(\nabla_U R)(X,Y,V,W) + (\nabla_V R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,V) = 0.$$

This is a standard trick for such calculations for tensors. To prove any relation at a $p \in M$, by multilinearity it suffices to prove when X, Y, U, V, W are basis elements w.r.t. some frame.

To prove this, let $p \in M$ be arbitrary and choose Riemannian normal coordinates centred at p. $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ is a local frame

$$g_{ij}(p) = \delta_{ij}$$

$$\left(\nabla_{\partial_i} \partial_j\right)_p = \Gamma_{ij}^{k}(p) \, \partial_k \Big|_p = 0$$

Let X, Y, U, V, W be $\partial_i, \partial_j, \partial_k, \partial_l, \partial_m$

now

$$(\nabla_U R)(X,Y,V,W) \underset{\text{defn}}{=} U(R(X,Y,V,W))$$

$$-R(\nabla_U X, Y, V, W)$$

$$- \dots - R(X, Y, V, \nabla_U W)$$

But $\nabla_U X, \dots, \nabla_U W = 0$ at p in normal coordinates

$\Rightarrow$ at p, $(\nabla_U R)(X, Y, V, W) = U(R(X, Y, V, W))$

now

$$U(R(X, Y, V, W)) = U(g(R(X, Y)V, W))$$

$$= U(g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \nabla_{[X,Y]} V, W))$$

$\nabla_{[X,Y]}$: $= 0$ as coordinate v.f

metric compatibility

$$= g(\nabla_U \nabla_X \nabla_Y V, W) - g(\nabla_U \nabla_Y \nabla_X V, W)$$

$$- g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V, \nabla_U W)$$

$\nabla_U W$: $= 0$ at p

$$\Rightarrow (\nabla_U R)(X, Y, V, W)(p) = U(R(X, Y, V, W))(p)$$

$$= U(R(V,W,X,Y),W)(p)$$

$$= g(\nabla_U \nabla_V \nabla_W X, Y)(p)$$
$$- g(\nabla_U \nabla_W \nabla_V X, Y)(p)$$

now cyclically permute U, V and W and then add to get the 2^{nd} Bianchi identity.

∎

Remark :- If d^∇ is the exterior covariant derivative then the 2^{nd} Bianchi identity is $d^\nabla R = 0$. (true for any connection on any vector bundle).

Other notions of curvature from Rm

Aside :- Let $(V, \langle , \rangle)$ be an IPS and

$\{e_1, \dots, e_n\}$ be a basis.

$A : V \longmapsto V$ be a linear map.

$Ae_i = A_i^j e_j$. Then $Tr(A) = A_i^i \in \mathbb{R}$.

Notice

$$g^{ij} \langle Ae_i, e_j \rangle = g^{ij} \langle A_i^l e_l , e_j \rangle$$
$$= g^{ij} A_i^l g_{lj} = A_i^i = tr(A)$$

Thus

$$\boxed{tr(A) = g^{ij} \langle Ae_i, e_j \rangle}$$

more generally if B_{ij} is a <u>bilinear form</u>

Define $\quad Tr_g(B) = g^{ij} B_{ij}$.

Let (M, g) Riemannian and fix $X_p, Y_p \in T_pM$

Define $\quad A_p : T_pM \longrightarrow T_pM$ be

$A_p(Z_p) = R(Z_p, X_p) Y_p$

$$Tr(A_p) = g(A_p e_i, e_j) g_p^{ij}$$

for <u>any</u> basis $e_1, \dots, e_n$ of T_pM.

$$= g(R(e_i, X_p)Y, e_j) g^{ij}$$

$$= R(e_i, X_p, Y_p, e_j) g^{ij}$$

<u>Defn</u> The <u>Ricci tensor</u> of g is the $(2,0)$ tensor Ric defined

$$Ric(X,Y) = g^{ij} R(e_i, X, Y, e_j)$$

for any local frame $\{e_1, \dots, e_n\}$

in local coordinates

$$Ric = R_{jk}\, dx^j \otimes dx^k \quad \text{where}$$

$$R_{jk} = R_{ijkl}\, g^{il}.$$

Remark :- Ricci is symmetric.

Claim:– The defⁿ $Ric(X,Y) = g^{ij} R(e_i, X, Y, e_j)$ is well-defined.

If $\tilde{e}_1, \dots, \tilde{e}_n$ be another local frame.

Then $\exists$ P s.t. $\tilde{e}_i = P_{im} e_m$

$\Rightarrow$ $\tilde{g}_{ij} = g(\tilde{e}_i, \tilde{e}_j) = P_{im} P_{jk} g_{mk}$

i.e. $\tilde{g} = P g P^T$, $\tilde{g}^{-1} = (P^T)^{-1} g^{-1} P^{-1}$

$\therefore$ $\tilde{g}^{il} R(\tilde{e}_i, X, Y, \tilde{e}_l) =$

$$(P^T)^{-1}_{ia} (g^{-1})_{ab} (P^{-1})_{bj} R(P_{ik} e_k, X, Y, P_{jm} e_m)$$

$$= (P^{-1})_{ai} (g^{-1})_{ab} (P^{-1})_{bj} P_{ik} P_{jm} R(e_k, X, Y, e_m)$$

$$= \delta_{ak} \delta_{bm} (g^{-1})_{ab} R(e_k, X, Y, e_m)$$

$$= g^{km} R(e_k, X, Y, e_m)$$

$\blacksquare$

Exercise:- Prove the previous remark.

What is the meaning of Ric?

Ric is determined by polarization from its associated quadratic form

$$q(X) = Ric(X, X).$$

Let $\{e_1, \ldots, e_n\}$ be a local o-n-frame

$$Ric(e_i, e_i) = g^{kl} R(e_k, e_i, e_i, e_l)$$

$$\underset{\text{o.n.}}{=} \sum_{k=1}^{n} R(e_k, e_i, e_i, e_k)$$

$$= \sum_{k \neq i}^{n} R(e_k, e_i, e_i, e_k)$$

$$|e_i|^2 |e_k|^2 \quad \langle e_i, e_k \rangle^2$$

[illegible]

$$= \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathcal{K}(e_k \wedge e_i)$$

$\mathcal{K}$ → sectional curvature

$e_k \wedge e_i$ → 2-plane spanned by e_k and e_i

Thus $Ric(e_i, e_i)$ is $(n-1)$ (average of all sectional curvatures of 2-planes containing e_i.)

Scalar Curvature

$$R = Tr_g(Ric) = g^{ij} R_{ij}$$

So R is a smooth function on M.

$R = n$ (average of Ricci curvature)

Special Cases :-

$n=1 \quad : \quad R_{ijkl} = 0$

$n=2 \quad : \quad$ Ricci $, \; R_{jk} = g^{il} R_{ijkl}$

$$R_{11} = g^{il} R_{i11l} = g^{22} R_{2112} = g^{22} R_{1221}$$

$$R_{22} = g^{il} R_{i22l} = g^{11} R_{1221} = g^{11} R_{1221}$$

$$R_{12} = g^{il} R_{i12l} = g^{12} R_{2121} = -g^{12} R_{1221}$$

Scalar $, \; R = g^{11} R_{11} + 2 g^{12} R_{12} + g^{22} R_{22}$

$$= 2\left(g^{11} g^{22} - (g^{12})^2\right) R_{1221}$$

$$= 2 R_{1221} \cdot \det(g^{-1})$$

$$= \frac{1}{\det(g)} 2 R_{1221} = 2K$$

$\therefore$ for $n=2$ $\boxed{R = 2K}$

<u>Defn</u> (M,g) is called <u>Einstein</u> if $\exists$ $\lambda \in C^{\infty}(M)$ s.t

$$\boxed{Ric = \lambda g}$$

Suppose (M, g) is Einstein. Then

$$R = g^{ij} R_{ij} = g^{ij} \lambda g_{ij} = n\lambda$$

$$\Rightarrow \quad \lambda = \frac{R}{n}$$

$$\therefore \quad \boxed{Ric = \frac{R}{n} g}$$

We'll see examples of Einstein metrics.

special case :- $Ric = 0$ or Ricci-flat.

<u>Aside</u> :- In GR, the natural equation is

$$\underbrace{Ric - \frac{R}{2} g}_{G} = T \text{ - prescribed RHS}$$

↳ stress-energy tensor

G = Einstein tensor

Suppose $T = 0 \Rightarrow Ric = R/_2 \, g$

tracing $\Rightarrow$

$$R = \frac{nR}{2} \Rightarrow n \neq 2 \Rightarrow R = 0 \text{ and}$$

$Ric = 0$.

$\therefore$ if $n > 2$ and $T = 0$ then M must be Ricci flat.

Exe. Prove the following :-

1. $\nabla_l R_{ljmk} = \nabla_k R_{jm} - \nabla_m R_{jk}$
2. $\mathrm{div}(Rc) = \frac{1}{2} dR$.

Lemma :- Diagonalize R on (M^3, g) w.r.t. basis $\{e_2 \wedge e_3, \, e_3 \wedge e_1, \, e_1 \wedge e_2\}$ of $\Lambda^2 TM^3$ w/ $\{e_1, e_2, e_3\}$ an o.n.b. of TM. Suppose that w.r.t. basis R is a diagonal matrix w/ entries $\lambda_1, \lambda_2, \lambda_3$. Then w.r.t. $\{e_1, e_2, e_3\}$ we have

$$Rc = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_3 & 0 & 0 \\ 0 & \lambda_3+\lambda_1 & 0 \\ 0 & 0 & \lambda_1+\lambda_2 \end{bmatrix}$$

and the scalar curvature $R = \lambda_1+\lambda_2+\lambda_3$.

Proof. Exercise

Lemma :- Let (M^n, g) be an Einstein manifold w/ $n \geq 3$. Then M has constant scalar curvature. If $n=3$ the g has constant sectional curvature.

Proof. exercise

Defn Constant curvature metrics.

$\mathbb{R}^n$ w/ Euclidean metric has constant sec. curvature 0.

$S^n_R = \{ x \in \mathbb{R}^{n+1}, |x| = R \}$ w/ the round metric has constant sectional curvature $\frac{1}{R^2}$.

H^n_R, the hyperbolic space of radius R which is an open ball of radius R in $\mathbb{R}^n$ w/ the metric

$$g_{ij}(x) = \frac{4R^4 \delta_{ij}}{\quad}$$

$$(R^2 - |x|^2)^2$$

has constant curvature $-1/R^2$.

Any complete, simply connected Riemm. n-fold w/ constant sectional curvature is isometric to one of the above depending on the sign.