The covariant derivative

To differentiate tensors we need a connection.

Defn:- Let $E \xrightarrow{\pi} M$ be a v.b. A connection on E is a map

$$\nabla : \Gamma(M) \times \Gamma(E) \to \Gamma(E) \text{ s.t.}$$

1) $\nabla_X Y$ is $C^\infty(M)$-linear in X.

2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y.

3) For $f \in C^\infty(M)$, ∇ satisfies the Leibniz rule

$$\nabla_X (fY) = X(f)Y + f\nabla_X Y.$$

$\nabla_X Y$ is the covariant derivative of Y in the direction of X.

∇ on E is completely determined by its Christoffel symbols Γ^k_{ij} which in local coordinates can be defined as

$$\nabla_{\partial_i} E_j = \Gamma^k_{ij} E_k .$$

Lemma:- If TM is the tangent bundle then we can define connections on all tensor bundles $T^k_l(M)$ s.t.

Check that the map $(X, Y) \longmapsto \nabla_X Y$ is not a tensor.

1. $\nabla_X f = X(f)$.
2. $\nabla_X (F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$.
3. $\nabla_X (\mathrm{tr}\, Y) = \mathrm{tr}(\nabla_X Y)$. for all traces over any index of Y.

In local coordinates

$$(\nabla_X F) = (\nabla_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k}) \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k} X^p$$

and also

$$\nabla_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k} = \partial_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k} + \sum_{s=1}^{l} F^{j_1 \cdots q \cdots j_l}_{i_1 \cdots i_k} \Gamma^{j_s}_{pq}$$
$$- \sum_{s=1}^{k} F^{j_1 \cdots j_l}_{i_1 \cdots q \cdots i_k} \Gamma^{q}_{p i_s}.$$

Defn **Gradient**

Let $f \in C^\infty(M)$. $df \in \Gamma(T^*M)$

$(df)^{\#} \in \Gamma(TM)$ is called the gradient of f w.r.t. g and is denoted by ∇f.

in local coordinates, $df = \frac{\partial f}{\partial x^j} dx^j$

$$(\nabla f) = (\nabla f)^i \frac{\partial}{\partial x^i}$$

$$= \left(g^{ij} \frac{\partial f}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

<u>example</u> S^2 w/ spherical coordinates.

round metric on S^2, $g = (d\phi)^2 + \sin^2\phi \, (d\theta)^2$

in these coordinates.

$$\nabla f = \frac{\partial f}{\partial \theta} g^{\theta\theta} \frac{\partial}{\partial \theta} + \frac{\partial f}{\partial \phi} g^{\phi\theta} \frac{\partial}{\partial \theta}$$

$$+ \frac{\partial f}{\partial \phi} g^{\theta\phi} \frac{\partial}{\partial \phi} + \frac{\partial f}{\partial \phi} g^{\phi\phi} \frac{\partial}{\partial \phi}$$

and $g\left(\frac{\partial}{\partial \phi}, \frac{\partial}{\partial \phi}\right) = 1$, $g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi}\right) = 0$

$$g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = \sin^2\phi$$

$$\therefore \nabla f = \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial f}{\partial \phi}\frac{\partial}{\partial \phi}$$

The Levi-Civita Connection

Let (M^n, g) Riemm. mfld.

Defn A connection ∇ on TM is said to be compatible with g if

$$\nabla g = 0.$$

(i.e. g is parallel)

If $\nabla g = 0 \Rightarrow \nabla_X g = 0 \quad \forall X$

$\Leftrightarrow (\nabla_X g)(Y, Z) = 0 \quad \forall Y, Z,$

$\Leftrightarrow X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$
$= 0$

In local coordinates,

$$\left(\nabla_{\frac{\partial}{\partial x^k}} g\right)_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}$$

$\therefore \quad \nabla g = 0 \Leftrightarrow$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} g_{lj} + \Gamma^l_{kj} g_{il} \quad \forall i, j, k$$

Recall :→ The torsion T^∇ of a connection ∇ on TM is

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Thm [Fundamental Theorem of Riemannian Geometry]

Let (M^n, g) be Riemm. Then $\exists!$ conne-ction ∇ that is both metric compatibl and torsion-free. ∇ is called the Levi-Civita connection.

Proof :→ We'll show that it must be unique if it exists. by deriving a formula for it (Koszul formula).

Let $X, Y, Z \in \Gamma(TM)$

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$Y(g(X,Z)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$Z(g(Y,X)) = g(\nabla_Z Y, X) + g(Y, \nabla_Z X)$$

and $\because$ $T^{\nabla} = 0$

$$\Rightarrow \quad \nabla_X Y - \nabla_Y X = [X,Y]$$

$$\nabla_Z X - \nabla_X Z = [Z,X]$$

$$\nabla_Y Z - \nabla_Z Y = [Y,Z]$$

$\therefore$ we get

$$X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))$$

$$= 2g(\nabla_X Y, Z) + g(Y,[X,Z]) + g(Z,[Y,X]) - g(X,[Z,Y])$$

$$\Rightarrow g(\nabla_X Y, Z) = \frac{1}{2}\Big[X(g(Y,Z)) + Y(g(X,Z)) + Z(g(X,Y)) - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])\Big]$$

So $\nabla_X Y$ is determined uniquely.

Define ∇ by this formula and show that ∇ is compatible and torsion free.

- in local coordinates, the Christoffel symbols of ∇^{LC} are [for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$]

$$\Gamma_{ij}^{m} g_{mk} = \frac{1}{2}\left[\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij}\right]$$

$$\boxed{\Rightarrow \Gamma_{ij}^{k} = \frac{1}{2} g^{kl}\left[\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right]}$$

We'll use this formula frequently.

<u>Def^n</u> If U and $V \subseteq \mathbb{R}^n$ are open then a diffeo $\psi: U \to V$ is called orientation preserving if the Jaco-
-bian matrix $D\psi(p) \in GL(n, \mathbb{R})$ has positive deter-
-minant $\forall\, p \in U$.

e.g. $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ is an orientation preserving diffeo.

<u>Def^n</u> A smooth atlas $\mathcal{A} = \{(U_\alpha, x_\alpha)_{\alpha \in I}\}$ is <u>oriented</u> if all of its transition maps $x_\alpha \circ x_\beta^{-1}$ are orientation preserving.

An orientation of M w/ maximal smooth atlas $\mathcal{A}$ is a subset $\mathcal{A}^+ \subset \mathcal{A}$ that forms a maximal oriented atlas for M.

A smooth manifold that has been equipped w/ an orientation $\mathcal{A}^+$ is called an <u>oriented manifold</u>.

Orientation

If M is orientable, then a choice of such a cover or equivalently, a choice of nowhere-zero n-form) is called an orientation for M.

Such a form μ is called a volume form on μ. Two volume forms $\mu, \tilde{\mu}$ corresponding to the same orientation $\Longleftrightarrow$ $\mu = f\,\tilde{\mu}$ for some $f \in C^\infty(M)$ s.t. f is everywhere positive.

Let M be orientable and have k-connected components then $\exists$ 2^k orientations on M.

If M^n is oriented, compact, we can integrate n-forms on M. $\int_M \omega \in \mathbb{R}$

$\omega \in \Omega^n(M)$

Stokes' Theorem If $\partial M = \phi$

then $\int_M d\sigma = 0$

If $F : M \xrightarrow{\text{diffeo}} N$

$\omega \in \Omega^n(N) \Rightarrow F^*\omega \in \Omega^n(M)$

$$\Rightarrow \boxed{\int_M F^*\omega = \int_{N=F(M)} \omega}$$

Defn :- A manifold w/ volume form is an oriented mfld M together w/ a particular choice μ (representative of the equivalence-class of the orientation).

If M is compact the we can integrate functions on M by defining

$$\int_M f := \int_M f\mu$$

whose value depends on the choice of μ

Let (M, μ) be a manifold w/ volume form

Define the divergence $\mathrm{div} : \Gamma(TM) \longrightarrow C^\infty(M)$
linear

by
$$\mathcal{L}_X \mu = d(X \lrcorner \mu) + \underbrace{X \lrcorner d\mu}_{=0}$$
$$= (\mathrm{div}\, X)\, \mu$$

(depends on μ)

<u>Notice</u> :- $\operatorname{div} X = 0 \iff \mathcal{L}_X \mu = 0$

$\iff \theta_t^* \mu = \mu$ where

θ_t is the flow of X.

$\iff \mu$ is invariant under flow of X.

If M compact,

$$\operatorname{vol}(M) = \int_M 1 = \int_M 1 \cdot \mu$$

Suppose $\operatorname{div} X = 0 \implies$

$$\int_{\theta_t(M)} \mu = \operatorname{vol}(\theta_t(M)) = \int_M \theta_t^* \mu = \int_M \mu = \operatorname{vol}(M)$$

$$\implies \operatorname{vol}(\theta_t(M)) = \operatorname{vol}(M)$$

$\therefore$ flow of a Divergence-free v.f. preserves the volume.

Divergence Theorem

Let $X \in \Gamma(TM)$, (M, μ) be compact

then $\int_M (\operatorname{div} X) = 0$ as

$$\int_M (\operatorname{div} X)\mu = \int_M d(X \lrcorner \mu) = 0 \quad \text{by Stokes' Thm.}$$

Let (M, g) be an oriented Riemannian manifold. Then $\exists$ a canonical volume form μ on (M, g) defined by the requirement that

$$\mu(e_1, \dots, e_n) = 1 \quad \text{whenever } \{e_1, \dots, e_n\}$$

is an oriented orthonormal basis of (T_pM, g_p).

i.e., given a local oriented o.n. frame for M $\{e_1, \dots, e_n\}$,

$$\mu = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

$$\mu = \sqrt{\det g}\, dx^1 \wedge \cdots \wedge dx^n$$ in any local coordinates $(x^1, x^2, \dots, x^n)$.

- Divergence theorem holds for any manifold w/ volume $\Rightarrow$ also holds for oriented Riemn. vol. form and symplectic manifolds.

Curvature of the Levi-Civita connection

We call R, as the Riemann curvature tensor of

g.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$= -R(Y,X)Z$$

Remark :- $R^{\nabla}=0$ and $T^{\nabla}=0$ iff $\exists$ local parallel coordinate frames.

check that
$$Rm(fX,Y)Z = Rm(X,fY)Z = Rm(X,Y)(fZ) = f\,Rm(X,Y)Z.$$

One defn of being flat for any connection is $R^{\nabla}=0$

and for a Riem. mfld we defined flat as "locally isometric" to $(\mathbb{R}^n, \hat{g})$.

For the Riemannian curvature of Levi-Civita conn, the two notions of flatness are the same.

If we define

$$\nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z \quad \text{then}$$

$$Rm(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z \,.$$

The components of the (3,1)-tensor Rm are defined as

$$Rm\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R_{ijk}^{\ \ \ \ l} \frac{\partial}{\partial x^l} \,.$$

we also define

$$R_{ijkl} = R_{ijk}^{\ \ \ \ m} g_{lm}$$ which gives the components of the (4,0)-Rm

$$R_{ijkl} = Rm\left(\partial_i, \partial_j, \partial_k, \partial_l\right) = \left\langle Rm(\partial_i, \partial_j)\partial_k, \partial_l \right\rangle .$$

Remark :- One must be careful and check the convention for lowering the upper index to the lower one.