

Problem Set 4
Due date: 26.06.2025

Problems

(1) Prove that the set
$$K = \{M \mid \mu(M) + \nu(M) \geq 0\} \tag{0.1}$$
with M a curvature-like tensor, ν being the smallest eigenvalue and μ being the middle eigenvalue, is preserved by the associated ODE. Conclude from this, using the maximum principle for systems, that non-negative Ricci curvature is preserved along the Ricci flow in dimension 3.

(2) Prove the following evolution along the Ricci flow in dimension 3:
$$\frac{\partial}{\partial t}\left(|\operatorname{Ric}|^2 - \frac{1}{3}R^2\right) = \Delta\left(|\operatorname{Ric}|^2 - \frac{1}{3}R^2\right) - 2\left(|\nabla \operatorname{Ric}|^2 - \frac{1}{3}|\nabla R|^2\right) - 8\operatorname{tr}(\operatorname{Ric}^3) + \frac{26}{3}R|\operatorname{Ric}|^2 - 2R^3. \tag{0.2}$$

(3) First prove that in dimension 3, $\left|\nabla \operatorname{Ric} - \frac{1}{3}\nabla R g\right|^2 \geq \frac{1}{3}\left|\operatorname{div}\left(\operatorname{Ric} - \frac{1}{3}Rg\right)\right|^2$. Then using this inequality and the contracted second Bianchi identity, prove that
$$|\nabla \operatorname{Ric}|^2 \geq \frac{37}{108}|\nabla R|^2. \tag{0.3}$$