Problem Set 3
Due date: 12.06.2025

Problems

(1) Use the maximum principle for tensors to prove that the following curvature conditions are preserved along the Ricci flow on a closed M^3.

(a) $\mathrm{Ric} \geq 0$ is the sense of bilinear forms.

(b) The pinching inequality $R_{ij} \geq \varepsilon R g_{ij}$ is preserved for all $\varepsilon \leq \frac{1}{3}$.

(c) $R_{ij} \leq \frac{1}{2} R g_{ij}$ is preserved along the RF.

(2) Let $(M^2, g(t))$ be a Ricci flow on a closed surface with positive curvature R and define

$$Q_{ij} = \nabla_i \nabla_j \log R + \frac{1}{2}\left(R + \frac{1}{t}\right) g_{ij}. \tag{0.1}$$

Prove that $Q_{ij} \geq 0 \; \forall \, t$. This is called the **matrix Harnack estimate for the Ricci flow on surfaces.** We saw a version of this result in the course last semester.

(3) Recall that for a Lie algebra $\mathfrak{g}$ with basis $\{\phi^\alpha\}$, a symmetric bilinear form $L \in \mathfrak{g} \otimes_S \mathfrak{g}$ can be described by its components $L_{\alpha\beta} = L(\phi^*_\alpha, \phi^*_\beta)$ and we get the operation # between two bilinear forms as

$$(L \# M)_{\alpha\beta} = C_\alpha^{\gamma\epsilon} C_\beta^{\delta\zeta} L_{\gamma\delta} M_{\epsilon\zeta}$$

where $C_\gamma^{\alpha\beta}$ are the structure constants defined by

$$\left[\phi^\alpha, \phi^\beta\right] = \sum_\gamma C_\gamma^{\alpha\beta} \phi^\gamma.$$

(a) Prove that $L \# M = M \# L$.
(b) Prove that if $L \geq 0$, then $L^\# = L \# L \geq 0$.