Problem Set 1
Due date: 22.05.2025

Problems

(1) Recall that the Riemann curvature tensor can be further decomposed as follows:

$$\mathrm{Rm} = \frac{R}{2n(n-1)}(g \odot g) + \frac{1}{n-2}(\mathring{\mathrm{Ric}} \odot g) + W \tag{0.1}$$

where R is the scalar curvture, $\mathring{\mathrm{Ric}}$ is trace-free Ricci tensor, $\mathring{\mathrm{Ric}}_{ij} = R_{ij} - \frac{R}{n} g_{ij}$, W is the $Weyl$ tensor and $\odot$ is the **Kulkarni–Nomizu** product, which is a product of symmetric 2-tensors and is given for symmetric 2-tensors P and Q by

$$(P \odot Q)_{ijkl} = P_{il}Q_{jk} + P_{jk}Q_{il} - P_{ik}Q_{jl} - P_{jl}Q_{ik}.$$

(If you have not seen (0.1) before, look up Ricci decompostion or decomposition of Riemann curvature.)

Fact: The Weyl tensor vanishes in dimension 3.

Use (0.1) to prove that in dimension 3, the Ricci curvature of a solution to the Ricci flow evolves as

$$\partial_t R_{jk} = \Delta R_{jk} + 3RR_{jk} - 6R_{jp}R_{pk} + (2|\mathrm{Ric}|^2 - R^2)g_{jk}. \tag{0.2}$$

(2) Let $(M^n, g(t))_{t\in[0,T]}$ be a solution of the Ricci flow. For a constant C, suppose we have the pointwise bound for the scalar curvature $R(x,0) \geq C$ for all $x \in M$. Prove that

$$\mathrm{Vol}(M, g(t)) \leq e^{-2Ct}\mathrm{Vol}(M, g(0))$$

for all $t \in [0,T]$.

(3) Let us define the tensor B on (M^n, g) by

$$B_{ijkl} = -R_{pij}{}^{q} R_{qlk}{}^{p}. \tag{0.3}$$

Prove that the Riemann curvature tensor satisfies the following evolution along the Ricci flow

$$\begin{aligned}\partial_t(R)_{ijkl} = {} & \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ & - (R_i^p R_{pjkl} + R_j^p R_{ipkl} + R_k^p R_{ijpl} + R_l^p R_{ijkp}).\end{aligned} \tag{0.4}$$

(4) Let $(M^n, g(t))$ be a closed manifold with $g(t)$ a Ricci flow on $[0,T]$. Prove that if $|\,\mathrm{Ric}\,| \leq K$ on $[0,T]$ for some constant K then all the metrics in the family $g(t)$ are uniformly equivalent, i.e., for all $x \in M$ and $t \in [0,T)$,

$$e^{-2KT}g(x,0) \leq g(x,t) \leq e^{2KT}g(x,0). \tag{0.5}$$