Problem Set 1
Due date: 08.05.2025

Problems

(1) Let M^n be a closed manifold.

(a) Prove the **Bochner formula** for $|\nabla f|^2$, i.e., for $f \in C^\infty(M)$, prove that
$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla^i f \nabla^j f + 2\nabla_i f \nabla^i(\Delta f). \tag{0.1}$$
Conclude from this that if $\mathrm{Ric} \geq 0$, $\Delta f = 0$ and $|\nabla f| =$ constant then ∇f is parallel.

(b) Prove the following integral equality:
$$\int_M |\nabla\nabla f|^2 \,\mathrm{vol} + \int_M \mathrm{Ric}(\nabla f, \nabla f)\,\mathrm{vol} = \int_M (\Delta f)^2 \,\mathrm{vol} \tag{0.2}$$
and using the fact that[1], $|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2$, show that
$$\int_M \mathrm{Ric}(\nabla f, \nabla f)\,\mathrm{vol} \leq \frac{n-1}{n}\int_M (\Delta f)^2 \,\mathrm{vol}. \tag{0.4}$$
(**Hint:** Integration by parts!)

(c) Use the above to prove the following theorem due to **Lichnerowicz.** Suppose f is an eigenfunction of Δ with eigenvalue $\lambda > 0$, i.e., $\Delta f + \lambda f = 0$. If $\mathrm{Ric} \geq (n-1)K$ for some constant $K > 0$ then $\lambda \geq nK$.

(2) (a) The purpose of this problem is to show that in dimension 3, the Ricci curvature determines the Riemann curvature tensor.
Let (M^3, g) be a 3-dimensional Riemannian manifold and let us diagonalize the curvature operator (as a self-adjoint operator on 2-forms) Rm with respect to a basis $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$ of $\Lambda^2 T^*M$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of TM^3 (this is possible because Rm is self-adjoint). Suppose that, with respect to this basis, Rm is a diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3$ down the diagonal. Then with respect to the basis $\{e_1, e_2, e_3\}$, prove that the Ricci tensor takes the form
$$\mathrm{Ric} = \frac{1}{2}\begin{bmatrix} \lambda_2 + \lambda_3 & 0 & 0 \\ 0 & \lambda_3 + \lambda_1 & 0 \\ 0 & 0 & \lambda_1 + \lambda_2 \end{bmatrix} \tag{0.5}$$
and the scalar curvature $R = \lambda_1 + \lambda_2 + \lambda_3$. (**Hint:** Use the geometric interpretation of the Ricci and scalar curvatures.)

(b) Prove that an Einstein metric on a manifold of dimension $n \geq 3$ has constant scalar curvature. If $n = 3$, the metric has constant sectional curvature.

(3) Instead of the Ricci flow, one can also look at the volume normalized version of the Ricci flow called the **normalized Ricci flow** which is the the following evolution equation for a family of metrics $g(t)$ on M^n:
$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(g(t)) + \frac{2}{n}\frac{(\int_M R\,\mathrm{vol})}{(\int_M \mathrm{vol})}g(t) \tag{NRF}$$
where R is the scalar curvature. The advantage of (NRF) is that the volume of the evolving manifolds remains constant along (NRF). Prove that:

(a) The volume of the manifold remains constant along the NRF.

(b) A compact manifold (M^n, g) is a fixed point of (NRF) if and only if it is an Einstein manifold.

(c) Show that the unnormalized and normalized Ricci flows differ only by a rescaling of space and time.

[1]This is the usual Cauchy-Schwarz inequality. More generally, if S is any $(2,0)-$tensor then
$$|S_{ij}|_g^2 \geq \frac{1}{n}(g^{ij}S_{ij})^2 \tag{0.3}$$