Lec. 2, 3 & 4 - Basics of Riemannian geometry

Def^n M^n is a n-manifold if it is Hausdorff and paracompact and $\forall\ p \in M\ \exists\ U \ni p$ open in M and a function $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$.

(U, φ) is called a coordinate chart.

we denote $\varphi(q) = (x^1(q), x^2(q), \dots, x^n(q))$ w/ $x^i(q)$ being referred to as local coordinates for M^n.

Paracompact -

a refinement of an open cover $\{U_\alpha\}_{\alpha \in I}$ is another open cover $\{V_\beta\}_{\beta \in J}$ s.t. $\forall\ \beta \in J$, $V_\beta \subset U_\alpha$ for some $\alpha \in I$.

A top. space X is paracompact if every open cover X admits a locally finite refinement, i.e., every point in X has a nbd that intersects at most finitely many of the sets from the refinement.

This is used in the existence of partition of unity which in turn is used in proving the existence of a Riemannian metric.

Def^n Let (U, φ) and (V, ψ) be two coordinate charts on M, $U \cap V \neq \phi$.

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$ is a transition map.

- M is smooth or C^∞ if all transition maps are smooth.
- M is orientable if all transition maps are orientation-preserving.

Def^n Let $f : M \longrightarrow N$ be a map b/w smooth manifolds. f is called smooth if for every pair of coordinate charts (U, φ) of M and (V, ψ) of N,

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \longrightarrow \psi(f(U) \cap V)$$

is smooth.

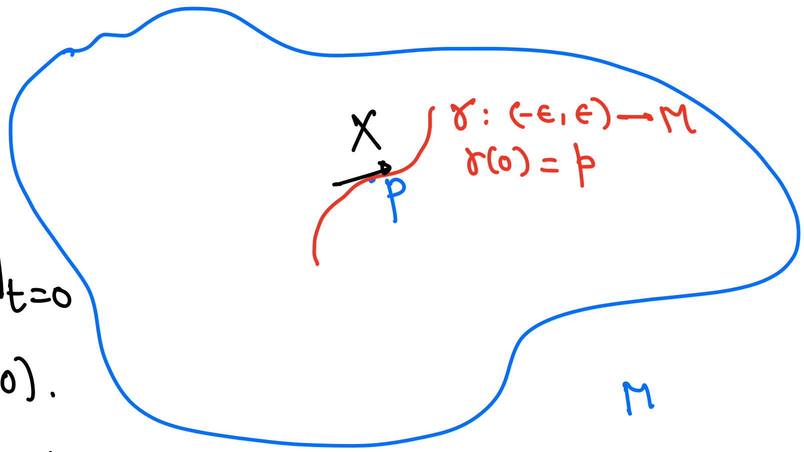
- $C^\infty(M) = \{ f : M \to \mathbb{R} \mid f \text{ is } C^\infty \}$.

Def^n:- Tangent vector X to M at $p \in M$ is a derivation i.e, X is an $\mathbb{R}$-linear function $X : C^\infty(M) \to \mathbb{R}$ which satisfies the Leibnitz rule

$$X(fg) = X(f)\, g(p) + f(p)\, X(g).$$

$T_pM^n = \{ X : X \text{ is a tangent vector to } M \text{ at } p \}$
is an n-dim $\mathbb{R}$-vector space.

Intuitively,



$$X(f) = \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$$

and then $X = \dot{\gamma}(0)$.

So X is indeed the "velocity vector".

If (x^i) is a local coordinate system then $\left\{ \frac{\partial}{\partial x^i}, i = 1, \dots, n \right\}$ forms a basis of T_pM^n. We'll often write ∂_i for $\frac{\partial}{\partial x^i}$.

The set of all tangent vectors at all points on M^n is itself a $2n$-dim manifold (in fact a vector bundle over M) called the tangent bundle of M TM.

Vector field X on M is a smoothly varying choice of tangent vector at each point $p \in M$, i.e., $\forall p \in M$, $X(p) \in T_pM^n$ and $X(f) \in C^\infty(M)$ $\forall f \in C^\infty(M)$.

Lie bracket $[X,Y]$ of two v.f. X and Y on M is again a vector field defined by

$$[X,Y]f = X(Y(f)) - Y(X(f)).$$

Def^n A rank k <u>vector bundle</u> $E \xrightarrow{\pi} M$ is given by the following: π is a surjective map called the projection map

- $\forall p \in M$, $E_p = \pi^{-1}(p)$ called the fibre of E over p is a k-dim. $\mathbb{R}$-v.s.
- $\forall p \in M$ $\exists$ an open nbd $U \ni p$ and a C^∞ diffeo $\varphi: \pi^{-1}(U) \to U \times \mathbb{R}^k$ s.t. φ takes each fibre E_p to $\{p\} \times \mathbb{R}^k$. This is called a <u>local trivialization</u>.

A section of E is a map $F: M \to E$ s.t. $\pi \circ F = id_M$. The space of sections of E will be denoted by either $\Gamma(E)$ or $C^\infty(E)$.

e.g. a v.f. $X \in \Gamma(TM)$.

we can also define the cotangent bundle T^*M whose fibres are $T_p^*M = (T_pM)^*$ is the dual space.

In coordinates (x^i) at p on M, $\{dx^i, i = 1, \dots, n\}$

w/ $dx^i(X) = X(x^i)$ forms a basis for T_p^*M.

Tensor bundles

We can take the usual tensor product of vector spaces and form the tensor bundles over M.

Let $V_1, \dots, V_n, W_1, \dots, W_m$ be $\mathbb{R}$-vector spaces. The <u>tensor product</u> $V_1 \otimes \cdots \otimes V_n \otimes W_1^* \otimes \cdots \otimes W_m^*$ is

the v.s. of multilinear maps $f: V_1^* \times V_2^* \times \cdots \times V_n^* \times W_1 \times \cdots \times W_m \to \mathbb{R}$.

A (p,q)-tensor field is a section of

$$T^p_q(M) = \underbrace{T^*M \otimes T^*M \otimes \cdots \otimes T^*M}_{p} \otimes \underbrace{TM \otimes TM \otimes \cdots \otimes TM}_{q}$$

If F is a (p,q) tensor and (x^i) is a coordinate system at $p \in M$ then we can express F in coordinates as

$$F = F^{j_1 \dots j_q}_{i_1 \dots i_p}(p)\, \partial_{j_1} \otimes \cdots \partial_{j_q} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_p}$$

w/ $F^{j_1 \dots j_q}_{i_1 \dots i_p} = F(\partial_{i_1}, \dots, \partial_{i_p}, dx^{j_1}, \dots, dx^{j_p})$.

We're using the Einstein summation convention, i.e., any index that is repeated twice, once lower and upper is being summed upon.

Given a tensor F, we can take the trace over one raised and one lowered index by defining

$$(\mathrm{tr}\,F)^{j_2\cdots j_q}_{i_2\cdots i_p} = F^{p\,j_2\cdots j_q}_{p\,i_2\cdots i_p} \in T^{p-1}_{q-1}(M).$$

(p is the index appearing over and under and thus the sum is over p).

A k-form ω is a section of $\Lambda^k T^*M$, i.e., it's a $(k,0)$ tensor field that is completely anti-symmetric

<u>**Defn**</u>:- Let A be a $(2,0)$-tensor. We say $A>0$ ($A\geq 0$) if $A(v,v)>0$ ($A(v,v)\geq 0$) $\forall\ v\in TM, v\neq 0$. i.e., at every $p\in M$, $\forall\ v_p\in T_pM$, $A_p(v_p,v_p)\in\mathbb{R}>0$ (≥ 0 resp.)

<u>**Defn**</u> A Riemannian metric g on M is a smoothly varying $(2,0)$-tensor which is an inner product on T_pM $\forall\ p$. Thus g is a symmetric $(2,0)$-tensor which is positive definite $\forall\ p\in M$.

In local coordinates, (x^i)

$$g = g_{ij}\, dx^i\otimes dx^j \quad w/$$

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ji}$$

↳ smooth functions on the domain U.

So for every $X \in T_pM$,

$$|X|_g^2 = g(X, X).$$

(M^n, g) is called a Riemannian manifold.

Def given (M, g) we can define the length of a curve $\gamma : [0,1] \to M$ by

$$\ell(\gamma) = \int_0^1 \sqrt{g(\dot\gamma(t), \dot\gamma(t))}\, dt$$

w/ $\dot\gamma(t) = \frac{d\gamma}{dt}$. Thus, we can define a metric d induced by g as

$$d(p,q) = \inf \left\{ \ell(\gamma) \;\middle|\; \gamma \text{ is a curve in } M \text{ joining } p \text{ and } q \right\}.$$

Similarly, $B(p, r) = \{ q \in M \mid d(p,q) < r \}$ is an open ball of radius r centred at p.

- If $i : L \longrightarrow M$ is an immersion then

i^*g is a metric on L if g is a metric on M.

Example:- $S^n \subseteq \mathbb{R}^{n+1}$

The inclusion map is an immersion.

Locally, in graph coordinates

$$i(u^1, \ldots, u^n) = \left(u^1, u^2, \ldots, u^n, \sqrt{1-|\bar{u}|^2}\right)$$

$$|\bar{u}|^2 < 1$$

$$\Rightarrow \quad i_* = \left[\begin{array}{c|c} \mathrm{Id} & \begin{matrix} x \\ x \\ x \end{matrix} \\ \hline x \quad x & x \end{array}\right]_{(n+1)\times n}$$

$\Rightarrow$ rank n $\Rightarrow$ injective $\Rightarrow$ i is an immersion.

$i^*\hat{g}$ = metric on S^n, called the round metric.

Exe. find the explicit expression of $i^*\tilde{g}$.

Def^n Let (M, g_M) and (N, g_N) be Riemannian manifolds. A map

$$F : (M, g_M) \longrightarrow (N, g_N) \text{ is}$$

called an isometry if

a) F is a diffeomorphism.

b) $F^* g_N = g_M$

Two Riem. manifolds are called isometric if $\exists$ an isometry b/w them.

Isometric manifolds are indistinguishable in terms of their Riemannian geometry.

Def^n (M, g_M) and (N, g_N) are locally isometric if and only if

$\forall\, p \in M$, $\exists$ $U \ni p$ open and

$$F : U \longrightarrow F(U) = V \text{ open in } N$$

s.t. F is an isometry of $(U, g_{M|U})$ onto $(V, g_{N|V})$.

There may not exist a global isometry

e.g. S^1 is locally isometric to $\mathbb{R}$ but not globally isometric.

More generally, T^n "flat torus" is locally isometric to $\mathbb{R}^n$. $\square$

Def^n (M^n, g) is called **flat** if it is locally isometric to $(\mathbb{R}^n, \hat{g})$.

Prop :- Let M^n be smooth. Then there are many Riemannian metrics on M.

Proof :- Let $\{U_\alpha, \alpha \in A\}$ be a locally finite open cover of M and let $\{\psi_\alpha, \alpha \in A\}$ be a partition of unity subordinate to this open cover.

On U_α, define a metric g_α by

$$g_\alpha = \delta_{ij}\, dx^i dx^j$$

(i.e., pullback by the coordinate chart the Euclidean metric $\mathbb{R}^n$)

Define $g = \sum_{\alpha \in A} \psi_\alpha g_\alpha$

and g is a Riemannian metric of M as a convex combination of positive definite bilinear forms is positive definite.

$\boxed{}$

Musical Isomorphisms

Linear algebra :-

Let V^n be a $\mathbb{R}$-v.s and V^* be its dual. Let g be a pos. def. bilinear form on V.

Define $\mu : V \longrightarrow V^*$

$v \longmapsto g(v, \cdot) \in V^*$ is a linear map.

$(\ker \mu) = 0 \Rightarrow \mu$ is an <u>isomorphism</u> as $\dim(V) = \dim(V^*)$.

Let (M^n, g) be Riemannian, then g_p induces an isomorphism $T_pU \underset{\flat}{\cong} T_p^*M$ called the musical isomorphisms

$$X_p \in T_pM \, , \quad (X_p)^\flat \in T_p^*M$$

$$(X_p)^\flat (Y_p) \underset{\text{def.}}{=} g_p(X_p, Y_p)$$

in local coordinates :- $X_p = X^i \dfrac{\partial}{\partial x^i}\Big|_p$

$$(X_p)^\flat = \underset{\to\, ?}{A_k}\, dx^k\Big|_p$$

If $Y_p = Y^j \frac{\partial}{\partial x^j}\Big|_p \Rightarrow (X_p^\flat)(Y_p) = A_k\, dx^k\left(Y^j \frac{\partial}{\partial x^j}\right)$

$$= A_k Y^k$$

$$= g(X_p, Y_p) = X^i Y^k g_{ik}$$

$$\Rightarrow \quad A_k = X^i g_{ik}$$

$\therefore$ if $X = X^i \frac{\partial}{\partial x^i}$ then

$$X^\flat = \underbrace{X^i g_{ik}}_{(X^\flat)_k} dx^k$$

The inverse of $\flat : T_pM \to T_p^*M$ is

$\# : T_p^*M \to T_pM$. $\alpha^k = \underbrace{g^{ki}}_{\text{inverse of } g_{ik}} \alpha_i$.

$\because$ g_{ij} is a pos. def. symmetric matrix $\forall p \in M$,

g^{ij} is just the inverse of the matrix.

Clearly $g^{ij} g_{jk} = \delta^i_k$.

The covariant derivative

To differentiate tensors we need a connection.

Defⁿ:- Let $E \xrightarrow{\pi} M$ be a v.b. A connection on E is a map

$$\nabla : \Gamma(M) \times \Gamma(E) \to \Gamma(E) \text{ s.t.}$$

1) $\nabla_X Y$ is $C^\infty(M)$-linear in X.

2) $\nabla_X Y$ is $\mathbb{R}$-linear in Y.

3) For $f \in C^\infty(M)$, ∇ satisfies the Leibniz rule

$$\nabla_X (fY) = X(f)Y + f\nabla_X Y.$$

$\nabla_X Y$ is the covariant derivative of Y in the direction of X.

∇ on E is completely determined by its Christoffel symbols Γ^k_{ij} which in local coordinates can be defined as

$$\nabla_{\partial_i} E_j = \Gamma^k_{ij} E_k.$$

Lemma:- If TM is the tangent bundle then we can define connections on all tensor bundles $T^k_l(M)$ s.t.

1. $\nabla_X f = X(f)$.

2. $\nabla_X(F \otimes G) = (\nabla_X F) \otimes G + F \otimes (\nabla_X G)$.

3. $\nabla_X(\operatorname{tr} Y) = \operatorname{tr}(\nabla_X Y)$. for all traces over any index of Y.

In local coordinates

$$(\nabla_X F) = (\nabla_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k}) \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k} X^p$$

and also

$$\nabla_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k} = \partial_p F^{j_1 \cdots j_l}_{i_1 \cdots i_k} + \sum_{s=1}^{l} F^{j_1 \cdots q \cdots j_l}_{i_1 \cdots i_k} \Gamma^{j_s}_{pq} - \sum_{s=1}^{k} F^{j_1 \cdots j_l}_{i_1 \cdots q \cdots i_k} \Gamma^{q}_{p i_s}.$$

Defn Gradient

Let $f \in C^\infty(M)$. $df \in \Gamma(T^*M)$

$(df)^\# \in \Gamma(TM)$ is called the gradient of f w.r.t. g and is denoted by ∇f.

in local coordinates, $df = \dfrac{\partial f}{\partial x^j} dx^j$

$$(\nabla f) = (\nabla f)^i \frac{\partial}{\partial x^i}$$

$$= \left(g^{ij} \frac{\partial f}{\partial x^j} \right) \frac{\partial}{\partial x^i}$$

<u>example</u> S^2 w/ spherical coordinates.

round metric on S^2, $g = (d\phi)^2 + \sin^2\phi \, (d\theta)^2$ in these coordinates.

$$\nabla f = \frac{\partial f}{\partial \theta} g^{\theta\theta} \frac{\partial}{\partial \theta} + \frac{\partial f}{\partial \phi} g^{\phi\theta} \frac{\partial}{\partial \theta}$$

$$+ \frac{\partial f}{\partial \phi} g^{\theta\phi} \frac{\partial}{\partial \phi} + \frac{\partial f}{\partial \phi} g^{\phi\phi} \frac{\partial}{\partial \phi}$$

and $\quad g\left(\dfrac{\partial}{\partial \phi}, \dfrac{\partial}{\partial \phi}\right) = 1, \quad g\left(\dfrac{\partial}{\partial \theta}, \dfrac{\partial}{\partial \phi}\right) = 0$

$$g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = \sin^2\phi$$

$$\therefore \nabla f = \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial f}{\partial \phi}\frac{\partial}{\partial \phi}$$

The Levi-Civita Connection

Let (M^n, g) Riemm. mfld.

__Def^n__ A connection ∇ on TM is said to be compatible with g if

$$\nabla g = 0.$$

(i.e. g is parallel)

If $\nabla g = 0 \Rightarrow \nabla_X g = 0 \quad \forall\ X$

$$\Leftrightarrow \quad (\nabla_X g)(Y, Z) = 0 \quad \forall\ Y, Z,$$

$$\Leftrightarrow \quad X\big(g(Y,Z)\big) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

In local coordinates,

$$\left(\nabla_{\frac{\partial}{\partial x^k}} g\right)_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^l_{ki}\, g_{ij} - \Gamma^l_{kj}\, g_{il}$$

$\therefore \quad \nabla g = 0 \Leftrightarrow$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki}\, g_{ij} + \Gamma^l_{kj}\, g_{il} \quad \forall\ i, j, k$$

<u>Recall</u> :→ The <u>torsion</u> T^∇ of a connection ∇ on TM is

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Thm [Fundamental Theorem of Riemannian Geometry]

Let (M^n, g) be Riemm. Then $\exists!$ connection ∇ that is both metric compatible and torsion-free. ∇ is called the Levi-Civita connection.

Proof :→ We'll show that it must be unique if it exists. by deriving a formula for it (Koszul formula).

Let $X, Y, Z \in \Gamma(TM)$

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$Y(g(X,Z)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$Z(g(Y,X)) = g(\nabla_Z Y, X) + g(Y, \nabla_Z X)$$

and $\because$ $T^{\nabla} = 0$

$$\Rightarrow \quad \nabla_X Y - \nabla_Y X = [X,Y]$$

$$\nabla_Z X - \nabla_X Z = [Z,X]$$

$$\nabla_Y Z - \nabla_Z Y = [Y,Z]$$

$\therefore$ we get

$$X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))$$

$$= 2g(\nabla_X Y, Z) + g(Y,[X,Z]) + g(Z,[Y,X]) - g(X,[Z,Y])$$

$$\Rightarrow g(\nabla_X Y, Z) = \frac{1}{2}\Big[X(g(Y,Z)) + Y(g(X,Z)) + Z(g(X,Y)) - g(Y,[X,Z]) - g(Z,[Y,X]) + g(X,[Z,Y])\Big]$$

So $\nabla_X Y$ is determined uniquely.

Define ∇ by this formula and show that ∇ is compatible and torsion free.

- in local coordinates, the Christoffel symbols of ∇^{LC} are $\left(\text{for } \begin{array}{l} X = \partial_i \\ Y = \partial_j \\ Z = \partial_k \end{array} \right)$

$$\Gamma^m_{ij}\, g_{mk} = \frac{1}{2}\left[\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ik} - \frac{\partial}{\partial x^k} g_{ij}\right]$$

$$\boxed{\Rightarrow \Gamma^k_{ij} = \frac{1}{2} g^{kl}\left[\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right]}$$

We'll use this formula frequently.

Orientation

If M is orientable, then a choice of such a cover or equivalently, a choice of nowhere-zero n-form) is called an *orientation* for M.

Such a form μ is called a volume form on M. Two volume forms $\mu, \tilde{\mu}$ corresponding to the same orientation $\Longleftrightarrow$ $\mu = f\,\tilde{\mu}$ for some $f \in C^{\infty}(M)$ s.t. f is everywhere positive.

Let M be orientable and have k-connect-
-ed components then $\exists\ 2^k$ orientations on M.

If M^n is oriented, compact, we can
integrate n-forms on M. $\int_M \omega \in \mathbb{R}$

$\omega \in \Omega^n(M)$

Stokes' Theorem If $\partial M = \phi$
then $\int_M d\sigma = 0$

If $F: M \xrightarrow{\text{diffeo}} N$

$\omega \in \Omega^n(N) \Rightarrow F^*\omega \in \Omega^n(M)$

$$\Rightarrow \boxed{\int_M F^*\omega = \int_{N=F(M)} \omega}$$

Defn :- A manifold w/ volume form is an oriented mfld M together w/ a particular choice μ (representative of the equivalence-class of the orientation).

If M is compact the we can integrate functions on M by defining

$$\int_M f := \int_M f\mu$$

whose value depends on the choice of μ

Let (M, μ) be a manifold w/ volume form
Define the divergence $\operatorname{div} : \Gamma(TM) \longrightarrow C^\infty(M)$
linear

by
$$\mathcal{L}_X \mu = d(X \lrcorner \mu) + \underbrace{X \lrcorner d\mu}_{=0}$$
$$= (\operatorname{div} X)\,\mu$$

(Depends on μ)

Notice :- $\operatorname{div} X = 0 \iff \mathcal{L}_X \mu = 0$

$\iff \theta_t^* \mu = \mu$ where

θ_t is the flow of X.

$\iff \mu$ is invariant under flow of X.

If M compact,

$$\operatorname{vol}(M) = \int_M 1 = \int_M 1 \cdot \mu$$

Suppose $\operatorname{div} X = 0 \implies$

$$\int_{\theta_t(M)} \mu = \operatorname{vol}(\theta_t(M)) = \int_M \theta_t^* \mu = \int_M \mu = \operatorname{vol}(M)$$

$\implies \operatorname{vol}(\theta_t(M)) = \operatorname{vol}(M)$

$\therefore$ flow of a Divergence-free v.f. preserves the volume.

Divergence Theorem

Let $X \in \Gamma(TM)$, (M, μ) be compact

then $\int_M (\operatorname{div} X) = 0$ as

$$\int_M (\operatorname{div} X)\mu = \int_M d(X \lrcorner \mu) = 0 \quad \text{by Stokes' Thm.}$$

Let (M, g) be an oriented Riemannian manifold. Then $\exists$ a canonical volume form μ on (M, g) Defined by the requirement that

$$\mu(e_1, \dots, e_n) = 1 \quad \text{whenever } \{e_1, \dots, e_n\}$$

is an oriented orthonormal basis of (T_pM, g_p).

i.e., given a local oriented o.n. frame for M $\{e_1, \ldots, e_n\}$,

$$\mu = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

$$\mu = \sqrt{\det g}\; dx^1 \wedge \cdots \wedge dx^n$$ in any local coordinates $(x^1, x^2, \ldots, x^n)$.

- Divergence theorem holds for any manifold w/ volume $\Rightarrow$ also holds for oriented Riemm. vol. form and symplectic manifolds.

Curvature of the Levi-Civita connection

We call R, as the Riemann curvature tensor of

g.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$= -R(Y,X)Z$$

Remark :- $R^{\nabla} = 0$ and $T^{\nabla} = 0$ iff $\exists$ local parallel coordinate frames.

One defn of being flat for any connection is $R^{\nabla} = 0$

and for a Riem. mfld we defined flat as "locally isometric" to $(\mathbb{R}^n, \hat{g})$.

For the Riemannian curvature of Levi-Civita conn, the two notions of flatness are the same.

Symmetries of R

$$R(X,Y,Z,W) := g(R(X,Y)Z, W)$$

$\downarrow$

(4,0) tensor obtained from (3,1) R by musical isomorphisms.

$$R(\partial_i, \partial_j)\partial_k = R_{ijk}^{\ \ \ \ l}\,\partial_l$$

$$R(\partial_i, \partial_j, \partial_k, \partial_m) = R_{ijkm}$$

$$\boxed{R_{ijkm} = R_{ijk}^{\ \ \ \ l}\, g_{lm}}$$

Prop:-

a) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

b) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

c) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W)$

$= 0$

d) $R(X, Y, Z, W) = R(Z, W, X, Y)$.

a) is always true and w/ b) allows us to see $R \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 T^*M)$ i.e, as a symmetric bilinear forms on the space of 2-forms.

b) follows from metric compatibility, $\nabla g = 0$

c) is true for any torsion free connection on TM. It is called the First Bianchi identity.

d) follows from a), b) and c).

Proof :- a) done

b) since $\nabla g = 0 \Rightarrow$

$$Y(g(Z, Z)) = 2\, g(\nabla_Y Z, Z)$$

$$X(Y(g(Z,Z))) = 2\, X(g(\nabla_Y Z, Z))$$

$$= 2g(\nabla_X \nabla_Y Z, Z) \longrightarrow (1)$$

$$+ 2g(\nabla_y Z, \nabla_x Z)$$

$$Y(X(g(Z,Z))) = 2g(\nabla_y Z, \nabla_x Z) + 2g(Z, \nabla_y \nabla_x Z) \quad - (2)$$

$$[X,Y](g(Z,Z)) = 2g(Z, \nabla_{[X,Y]} Z) \quad - (3)$$

(1) + (2) − (3)

$$X(Y(g(Z,Z))) - Y(X(g(Z,Z))) - [X,Y](g(Z,Z)) = 0$$

$$= 2R(X,Y,Z,Z) = 0$$

$\Rightarrow$ polarize to get (b).

c) Want to show that

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

expand and use torsion-free. and use

Jacobi identity for $[\cdot,\cdot]$.

d) Write identity c) in 4 ways.

Sectional Curvature

Let (M,g) be Riemann.

Given $X_p, Y_p \in T_pM$

$$|X_p \wedge Y_p|^2_{g_p} = |X_p|^2_{g_p} |Y_p|^2_{g_p} - g_p(X_p, Y_p)^2$$

$$|X \wedge Y|^2 = |X|^2 |Y|^2 - \langle X, Y \rangle^2$$

<u>Defⁿ</u>:- Let L_p be a <u>2-dimensional</u> subspace of T_pM $(n \geq 2)$. Define the <u>sectional curvature</u> $K_p(L_p)$ of (M,g) at <u>p</u> in

"L_p direction" by

$$K_p(L_p) = \frac{R(X_p, Y_p, Y_p, X_p)}{|X_p \wedge Y_p|^2}$$

for any basis X_p, Y_p of L_p.

(denom. not zero as X_p, Y_p are basis).

if $\tilde{X} = aX + bY$

$\tilde{Y} = cX + dY$

$$\tilde{X} \wedge \tilde{Y} = (ad - bc)\, X \wedge Y$$

show that

$$\frac{R(\tilde{X}, \tilde{Y}, \tilde{Y}, \tilde{X})}{|\tilde{X} \wedge \tilde{Y}|^2} = \frac{R(X, Y, Y, X)}{|X \wedge Y|^2}$$

if $n = 2$, $L_p = T_pM \quad \forall p \in M$

$\Rightarrow$ sectional curvature is just a smooth function on M.

Lemma:- The sectional curvature determines the Riemann curvature and vice-versa.

Precisely, suppose V^n $(n \geq 2)$ is a $\mathbb{R}$-inner product space and R and $\tilde{R}$ be two trilinear maps s.t.

$\langle R(x,y,z), w \rangle$ and $\langle \tilde{R}(x,y,z), w \rangle$ are skew in x,y, skew in y,z and satisfy 1st Bianchi identity.

Let $x,y \in V$ be linearly independent.

Let $\sigma = \operatorname{span}\{x,y\}$

Define $$K(\sigma) = \frac{\langle R(x,y,y), x \rangle}{|x \wedge y|^2}$$

$$\tilde{K}(\sigma) = \langle \tilde{R}(x,y,y), x \rangle$$

$$\overline{\qquad\qquad\qquad}$$
$$|x\wedge y|^2$$

if $K = \tilde{K}$ $\forall\ \sigma \subseteq V$ then $R = \tilde{R}$.

<u>Lemma</u> Let V be a real vector space w/ $\dim V \geq 2$ and R and $\tilde{R}$ be trilinear maps $V\times V\times V \to V$ satisfying

$$\langle R(x,y,z), w\rangle = (x,y,z,w)$$
$$\langle \tilde{R}(x,y,z), w\rangle = (x,y,z,w)^{\sim}$$

have the following symmetries :-

$$(x,y,z,w) = -(y,x,z,w) = -(x,y,w,z)$$
$$= (z,w,x,y)$$

and $(x,y,z,w) + (y,z,x,w) + (z,x,y,w) = 0$

same for $\sim$.

Let x, y be linearly independent. Let $\sigma = \operatorname{span}\{x,y\}$

Define $$K(\sigma) = \frac{(x,y,y,x)}{|x\wedge y|^2}$$

$$\tilde{K}(\sigma) = \frac{(x, y, y, x)^{\sim}}{|x \wedge y|^2}$$

If $\tilde{K}(\sigma) = K(\sigma)$ $\forall$ 2-dimensional subspace $\sigma \subseteq V$ then $R = \tilde{R}$.

Proof By hypo. $(x, y, y, x) = (x, y, y, x)^{\sim}$ $\forall x, y$.

polarize $(x+y, z, z, x+y) = (x+y, z, z, x+y)^{\sim}$

$$\Rightarrow (x, z, z, y) + (y, z, z, x) = (x, z, z, y)^{\sim} + (y, z, z, x)^{\sim}$$

$$\Rightarrow 2(x, z, z, y) = 2(x, z, z, y)^{\sim}$$

By symmetries

$$\Rightarrow (x, z, z, y) = (x, z, z, y)^{\sim}$$

polarize again, $z \longmapsto z + w$

$$(x, z, w, y) + (x, w, z, y) = (x, z, w, y)^{\sim} + (x, w, z, y)^{\sim}$$

$$\Rightarrow \underbrace{(X,Z,W,Y) - (X,Z,W,Y)^\sim}_{} =$$

$$-(X,W,Z,Y) + (X,W,Z,Y)^\sim$$

$$= (W,X,Z,Y) - (W,X,Z,Y)^\sim$$

$\Rightarrow$ $\sim$ is invariant under cyclic permutation

$$X \mapsto Z \mapsto W \mapsto X$$

$$\Rightarrow \sum_{\substack{X,Y,Z \\ \text{cyclic}}} (X,Z,W,Y) - (X,Z,W,Y)^\sim$$

$$= 3(X,Z,W,Y) - 3(X,Z,W,Y)^\sim$$

$$\Rightarrow 0 - 0 = 0$$

$$\Rightarrow (X,Z,W,Y) = (X,Z,W,Y)^\sim$$

$\boxed{\text{ne}}$

2nd Bianchi Identity

Let (M,g) be Riemannian and R be the

Riemannian $(4,0)$ tensor. Then

$$(\nabla_U R)(X, Y, V, W) + (\nabla_V R)(X, Y, W, U) + (\nabla_W R)(X, Y, U, V) = 0.$$

To prove this, let $p \in M$ be arbitrary and choose Riemannian normal coordinates centred at p. $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ is a local frame

$$g_{ij}(p) = \delta_{ij}$$

$$\left(\nabla_{\partial_i} \partial_j \right)_p = \Gamma^k_{ij}(p)\, \partial_k \Big|_p = 0$$

Let X, Y, U, V, W be $\partial_i, \partial_j, \partial_k, \partial_l, \partial_m$

now

$$(\nabla_U R)(X, Y, V, W) \underset{\text{defn}}{=} U(R(X, Y, V, W))$$

$$-R(\nabla_u X, Y, V, W)$$

$$- \cdots - R(X, Y, V, \nabla_u W)$$

But $\nabla_u X, \ldots, \nabla_u W = 0$ at p in normal coordinates

$\Rightarrow$ at p, $(\nabla_u R)(X, Y, V, W) = U(R(X, Y, V, W))$

now

$$U(R(X, Y, V, W)) = U(g(R(X, Y)V, W))$$

$$= U(g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V - \underbrace{\nabla_{[X,Y]}}_{\substack{=0 \text{ as coordinate} \\ \text{v.f}}} V, W))$$

metric compatibility

$$\overset{\uparrow}{=} g(\nabla_u \nabla_X \nabla_Y V, W) - g(\nabla_u \nabla_Y \nabla_X V, W)$$

$$- g(\nabla_X \nabla_Y V - \nabla_Y \nabla_X V, \underbrace{\nabla_u W}_{=0 \text{ at } p})$$

$$\Rightarrow (\nabla_u R)(X, Y, V, W)(p) = U(R(X, Y, V, W))(p)$$

$$= U(R(V,W,X,Y),W)(p)$$

$$= g(\nabla_U \nabla_V \nabla_W X, Y)(p)$$
$$- g(\nabla_U \nabla_W \nabla_V X, Y)(p)$$

now cyclically permute U, V and W and then add to get the 2nd Bianchi identity.

$\boxtimes$

Remark :- If d^∇ is the exterior covariant derivative then the 2nd Bianchi identity is $d^\nabla R = 0$. (true for any connection on any vector bundle).

Other notions of curvature from Rm

Aside :- Let $(V, \langle , \rangle)$ be an IPS and $\{e_1, \dots, e_n\}$ be a basis.

$A : V \longmapsto V$ be a linear map.

$Ae_i = A_i^j e_j$. Then $Tr(A) = A_i^i \in \mathbb{R}$.

Notice

$$g^{ij} \langle Ae_i, e_j \rangle = g^{ij} \langle A_i^l e_l, e_j \rangle$$
$$= g^{ij} A_i^l g_{lj} = A_i^i = tr(A)$$

Thus

$$\boxed{tr(A) = g^{ij} \langle Ae_i, e_j \rangle}$$

more generally if B_{ij} is a bilinear form

Define $\quad Tr_g(B) = g^{ij} B_{ij}$.

Let (M, g) Riemannian and fix $X_p, Y_p \in T_pM$

Define $\quad A_p : T_pM \longrightarrow T_pM$ be

$$A_p(Z_p) = R(Z_p, X_p) Y_p$$

$$\mathrm{Tr}(A_p) = g(A_p e_i, e_j)\, g_p^{ij}$$

for any basis $e_1, \dots, e_n$ of T_pM.

$$= g(R(e_i, X_p)Y, e_j)\, g^{ij}$$

$$= R(e_i, X_p, Y_p, e_j)\, g^{ij}$$

Defn The Ricci tensor of g is the $(2,0)$ tensor Ric defined

$$\mathrm{Ric}(X,Y) = g^{ij} R(e_i, X, Y, e_j)$$

for any local frame $\{e_1, \dots, e_n\}$

in local coordinates

$$\mathrm{Ric} = R_{jk}\, dx^j \otimes dx^k \quad \text{where}$$

$$R_{jk} = R_{ijkl}\, g^{il}.$$

Remark :- Ricci is symmetric.

Exercise:- Prove the previous remark.

What is the meaning of Ric?

Ric is determined by polarization from its associated quadratic form

$$q(X) = Ric(X, X).$$

Let $\{e_1, \dots, e_n\}$ be a local o-n-frame

$$Ric(e_i, e_i) = g^{kl} R(e_k, e_i, e_i, e_l)$$

$$\underset{\text{o.n.}}{=} \sum_{k=1}^{n} R(e_k, e_i, e_i, e_k)$$

$$= \sum_{k \neq i}^{n} R(e_k, e_i, e_i, e_k)$$

$$|e_i|^2 |e_k|^2 - \langle e_i, e_k \rangle^2$$

$$= \sum_{\substack{k=1 \\ k \neq i}}^{n} \underbrace{\mathcal{K}}_{\text{sectional curvature}} (\underbrace{e_k \wedge e_i}_{\text{2-plane spanned by } e_k \text{ and } e_i})$$

Thus $Ric(e_i, e_i)$ is $(n-1)$ (average of all sectional curvatures of 2-planes containing e_i.)

Scalar Curvature

$$R = Tr_g(Ric) = g^{ij} R_{ij}$$

So R is a smooth function on M.

$R = n$ (average of Ricci curvature)

Special Cases :-

$n=1$: $R_{ijkl} = 0$

$n=2$: Ricci , $R_{jk} = g^{il} R_{ijkl}$

$$R_{11} = g^{il} R_{i11l} = g^{22} R_{2112} = g^{22} R_{1221}$$

$$R_{22} = g^{il} R_{i22l} = g^{11} R_{1221} = g^{11} R_{1221}$$

$$R_{12} = g^{il} R_{i12l} = g^{12} R_{2121} = -g^{12} R_{1221}$$

Scalar , $R = g^{11} R_{11} + 2g^{12} R_{12} + g^{22} R_{22}$

$$= 2\left(g^{11} g^{22} - (g^{12})^2\right) R_{1221}$$

$$= 2 R_{1221} \cdot \det(g^{-1})$$

$$= \frac{1}{\det(g)} 2 R_{1221} = 2K$$

$\therefore$ for $n=2$ $\boxed{R = 2K}$

Defn (M, g) is called Einstein if $\exists$ $\lambda \in C^\infty(M)$ s.t

$$\boxed{Ric = \lambda g}$$

Suppose (M, g) is Einstein. Then

$$R = g^{ij} R_{ij} = g^{ij} \lambda g_{ij} = n\lambda$$

$$\Rightarrow \quad \lambda = \frac{R}{n}$$

$$\therefore \quad \boxed{Ric = \frac{R}{n} g}$$

We'll see examples of Einstein metrics.

Special case :- $Ric = 0$ or Ricci-flat.

<u>Aside</u> :- In GR, the natural equation is

$$\underbrace{Ric - \frac{R}{2} g}_{G} = T \text{ - } \underbrace{\text{prescribed}}\ \text{RHS}$$

↳ stress-energy tensor

G = Einstein tensor

Suppose $T = 0 \Rightarrow \quad Ric = R/_2 \, g$

tracing $\Rightarrow$

$$R = \frac{nR}{2} \Rightarrow n \neq 2 \Rightarrow$$

$R = 0$ and

$Ric = 0$.

$\therefore$ if $n > 2$ and $T = 0$ then M must be Ricci flat.

Exe. Prove the following:-

1. $\nabla_\ell R_{\ell jmk} = \nabla_k R_{jm} - \nabla_m R_{jk}$
2. $\operatorname{div}(Rc) = \frac{1}{2} dR$.

Lemma:- Diagonalize R on (M^3, g) w.r.t. basis $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$ of $\Lambda^2 TM^3$ w/ $\{e_1, e_2, e_3\}$ an o.n.b. of TM. Suppose that w.r.t. basis R is a diagonal matrix w/ entries $\lambda_1, \lambda_2, \lambda_3$. Then w.r.t. $\{e_1, e_2, e_3\}$ we have

$$Rc = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_3 & 0 & 0 \\ 0 & \lambda_3+\lambda_1 & 0 \\ 0 & 0 & \lambda_1+\lambda_2 \end{bmatrix}$$

and the scalar curvature $R = \lambda_1+\lambda_2+\lambda_3$.

Proof. Exercise

Lemma :- Let (M^n, g) be an Einstein manifold w/ $n \geq 3$. Then M has constant scalar curvature. If $n=3$ the g has constant sectional curvature.

Proof. exercise

Defn Constant curvature metrics.

$\mathbb{R}^n$ w/ Euclidean metric has constant sec. curvature 0.

$S^n_R = \{ x \in \mathbb{R}^{n+1}, |x| = R \}$ w/ the round metric has constant sectional curvature $\frac{1}{R^2}$.

H^n_R, the hyperbolic space of radius R which is an open ball of radius R in $\mathbb{R}^n$ w/ the metric

$$g_{ij}(x) = \frac{4R^4 \delta_{ij}}{}$$

$$(R^2 - |x|^2)^2$$

has constant curvature $-1/R^2$.

Any complete, simply connected Riemm. n-fold w/ constant sectional curvature is isometric to one of the above depending on the sign.