Recall that the time derivative $\frac{\partial}{\partial t} g(t)$ is defined as

$$\left(\frac{\partial}{\partial t} g\right)(X, Y) = \underbrace{\frac{\partial}{\partial t} g(X, Y)}$$

time-derivative of the smooth function $g(X, Y)$.

In local coordinates,

$$g(t) = g_{ij}(t)\, dx^i \otimes dx^j$$

$$\Rightarrow \quad \partial_t g(t) = \dot{g}_{ij}(t)\, dx^i \otimes dx^j .$$

$\therefore$ the time derivative of the metric is the time derivative of its components functions w.r.t. a fixed basis.

Similarly $\left(\frac{\partial}{\partial t} \nabla\right)(X, Y) = \frac{\partial}{\partial t} \nabla_X Y$.

now ∇ is NOT tensorial, but $\frac{\partial}{\partial t} \nabla$ is a tensor as

$$(\partial_t \nabla)(X, fY) = \frac{\partial}{\partial t}(\nabla_X fY) = \frac{\partial}{\partial t}\left(X(f)Y + f\nabla_X Y\right)$$

$$= f\,\partial_t \nabla_X Y = f\left(\frac{\partial}{\partial t}\nabla\right)(X,Y) \qquad \boxed{\,}$$

Variational Formulas

Given any smooth family of metrics it is desirable to compute the variations of all the associated quantities. We summarize them below for the case when

$$\partial_t g_{ij} = h_{ij}\,, \qquad h \in \Gamma(T^*M \otimes_S T^*M).$$

Lemma :-

$\left(g^{ij}g_{jk} = \delta^i_k \Rightarrow (\partial_t g^{ij})g_{jk} = -g^{ij}h_{jk}\right)$

- $$\partial_t g^{ij} = -g^{ik}g^{jl}h_{kl}$$

- $$\partial_t \Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}\right)$$

- $$\partial_t R_{ijk}^{\ \ \ \ l} = \frac{1}{2}g^{lp}\left\{\begin{aligned} &\nabla_i\nabla_j h_{kp} + \nabla_i\nabla_k h_{jp} \\ &- \nabla_i\nabla_p h_{jk} - \nabla_j\nabla_i h_{kp} \\ &- \nabla_j\nabla_k h_{ip} + \nabla_j\nabla_p h_{ik}\end{aligned}\right\}$$

$$= \frac{1}{2} g^{lp} \left\{ \begin{array}{l} \nabla_i \nabla_k h_{jp} + \nabla_j \nabla_p h_{ik} - \nabla_i \nabla_p h_{jk} \\ - \nabla_j \nabla_k h_{ip} - R_{ijk}^{\ \ q} h_{qp} - R_{ijp}^{\ \ q} h_{kq} \end{array} \right\}$$

- $$\partial_t R_{jk} = \frac{1}{2} g^{pq} \left(\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp} \right)$$

- $$\partial_t R = -\Delta(\operatorname{tr} h) + \nabla^p \nabla^q h_{pq} - \langle h, Rc \rangle$$

- $$\partial_t \, vol_g = \frac{\operatorname{tr} h}{2} vol \quad \text{(proof below)}$$

- $$\partial_t \int R \, vol_g = \int_M \left(\frac{R(\operatorname{tr} h)}{2} - \langle h, Rc \rangle \right) vol$$

Along the <u>RF</u> we have following improvements

$$\partial_t R = \Delta R + 2|Rc|^2 \text{ — proof below.}$$

$$\partial_t R_{jk} = \Delta R_{jk} + 2 g^{pq} g^{rs} R_{pjkr} R_{qs} - 2 g^{pq} R_{jp} R_{qk}.$$

$\downarrow$

proof:- $\partial_t R_{jk} = \Delta R_{jk} + \nabla_j \nabla_k R - g^{pq}\big(\nabla_q \nabla_j R_{kp} + \nabla_q \nabla_k R_{jp} \big)$

$= \Delta R_{jk} + \nabla_j \nabla_k R - g^{pq}\big(\nabla_j \nabla_q R_{kp} - R_{qjkm} R_{mp} - R_{qjpm} R_{km} + \nabla_k \nabla_q R_{jp} - R_{qkjm} R_{mp} - R_{qkpm} R_{jm} \big)$

$= \Delta R_{jk} + \nabla_j \nabla_k R - \big(\frac{1}{2} \nabla_j \nabla_k R + \frac{1}{2} \nabla_k \nabla_j R - R_{pjkm} R_{pm} + R_{jm} R_{km} - R_{pkjm} R_{pm} + R_{km} R_{jm} \big)$

$=$ RHS.

<u>Proof for $\partial_t R$.</u>

we have $\partial_t R = -\Delta(\mathrm{tr}(-2Rc)) + \mathrm{div}(\mathrm{div}(-2Rc)) - \langle -2Rc, Rc \rangle$

$= 2\Delta R - \Delta R + 2|Rc|^2$ (we use twice contracted 2nd Bianchi)

$= \Delta R + 2|Ric|^2.$

Proof for the evolution of vol.

First recall that in local coordinates, the volume form

$$\mathrm{vol}_g = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^n$$

↳ det of $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

Recall that

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A \quad \text{for a square matrix } A$$

where $\operatorname{adj} A$ = adjugate matrix = transpose of the cofactor matrix

The partial derivative of $\det A$ w.r.t. (i,j)-th entry is

$$\frac{\partial}{\partial a_{ij}} \det(A) = (-1)^{i+j} \det A_{ij}$$

$$= (\operatorname{adj} A)_{ji} = \det A \, (A^{-1})_{ji}$$

So

$$\frac{\partial}{\partial t} \sqrt{\det g_{ij}} = \frac{1}{2\sqrt{\det g_{ij}}} \frac{\partial}{\partial t} \det g = \frac{1}{2\sqrt{\det g}} \frac{\partial \det g}{\partial g_{ij}} \frac{\partial g_{ij}}{\partial t}$$

$$= \frac{1}{2\sqrt{\det g}} \det g \, (g^{-1})_{ji} \, h_{ij}$$

$$= \frac{1}{2} \sqrt{\det g} \; g^{ij} h_{ij}$$

$$\therefore \; \partial_t \mathrm{vol} = \frac{(\mathrm{tr}\, h)}{2} \mathrm{vol} .$$

The proofs for the evolutions of Rm, Ric, R and Γ for general variations can be done using the local coordinate expressions of these quantities and noticing that they are all components of a tensor (Γ is not but $\partial_t \Gamma$ is) and hence we can simplify our calculations by working in normal coordinates at a point.

We did this in detail in the class.

——— x ——— x ———