DeTurck's trick

Recall:-

- RF is NOT parabolic.
- Diffeo invariance is the only obstruction to the parabol. of RF.

Theorem (Hamilton '82)

If (M^n, g_0) is a closed manifold then $\exists !$ $g(t)$ defined for $t \in [0, \epsilon)$ to RF s.t.

$g(0) = g_0$, $\epsilon > 0$.

$g_1(t)$, $g_2(t)$, $g_1(0) = g_2(0)$ then

$g_1(t) = g_2(t)$ $\forall t$.

$g_1(s) = g_2(s)$ for some $s \in \mathbb{R}$.

$\Rightarrow g_1(t) = g_2(t)$ $\forall t$.

DeTurck's trick

$$-2[D(Ric_g)(h)]_{ik} = \Delta h_{ik} + g^{pq}\big(\nabla_i \nabla_k h_{pq} - \nabla_q \nabla_i h_{kp} - \nabla_q \nabla_k h_{ip}\big)$$

$$= \Delta h_{ik} + g^{pq}\big(\nabla_i \nabla_k h_{pq} - \nabla_i \nabla_q h_{kp} + R_{qiks} h_{sp} + R_{qips} h_{ks} - \nabla_k \nabla_q h_{ip} + R_{qkis} h_{sp} + R_{qkps} h_{is}\big)$$

$$= \Delta h_{ik} + g^{pq} \nabla_i \nabla_k h_{pq} - g^{pq} \nabla_i \nabla_q h_{pk} - g^{pq} \nabla_k \nabla_q h_{pi} + S_{ik} \longrightarrow (1)$$

$$B_g : \Gamma(S^2 T^* M) \to \Gamma(T^* M)$$

$$[B_g(h)]_k = \nabla_i h_{ik} - \frac{1}{2} \nabla_k \operatorname{tr} h$$

Suppose $v = B_g(h)$

$$v_k = g^{pq}\left(\nabla_q h_{pk} - \frac{1}{2} \nabla_k h_{pq}\right)$$

$$\nabla_i v_k = g^{pq}\left(\nabla_i \nabla_q h_{pk} - \frac{1}{2} \nabla_i \nabla_k h_{pq}\right)$$

$$\nabla_k v_i = g^{pq}\left(\nabla_k \nabla_q h_{pi} - \frac{1}{2} \nabla_k \nabla_i h_{pq}\right) \longrightarrow (2)$$

from (1) and (2) we see that

$$-2\,[D(\mathrm{Ric}_g)(h)]_{ik} = \Delta h_{ik} - \nabla_i v_k - \nabla_k v_i + S_{ik}. \longrightarrow (3)$$

$$V_k = \frac{1}{2} g^{pq} \left(\nabla_p h_{qk} + \nabla_q h_{pk} - \nabla_k h_{pq} \right)$$

$$= g^{pq} g_{k\gamma} \left(D\Gamma_g(h) \right)^{\gamma}_{pq}$$

fix some background metric $\tilde{g}$ on M, L-C connection $\tilde{\Gamma}$

define a vector field W

$$W^k = g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq} \right)$$

W is well-defined v.f. on M as difference of two connections is a tensor.

Look at $P = P(\tilde{\Gamma}) : \Gamma(S^2 T^* M) \to \Gamma(S^2 T^* M)$

$$P(g) = \mathcal{L}_W g$$

P is a 2nd-order operator.

$$[DP(h)]_{ik} = \nabla_i V_k + \nabla_k V_i + \text{l.o.t.}$$

$\longrightarrow$ 4

We look at the modified operator

$$Q = -2Ric + P$$

$DQ(h) = \Delta h + \text{l.o.t.}$

Ricci-DeTurck flow

$\therefore$ Q is elliptic $\Rightarrow$ $\overbrace{\partial_t g = Q(g)}$ is parabolic $\Rightarrow$ $\exists!$ solution to

$\partial_t g = -2Ric(g) + P(g)$ for some short time.

Relate sol^n of the R-DT flow to solutions of the RF

$$\partial_t g_{ij} = -2R_{ij} + \nabla_i W_j + \nabla_j W_i$$

$$g(0) = g_0$$

— RDTF

$$W_j = g_{jk}\, g^{pq}\left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}\right)$$

has a sol^n $g(t)$, $t \in [0, \epsilon)$.

$\therefore$ sol^n to RDTF exists $\Rightarrow$ $\exists$ family of v.f. $W(t)$ exists $\forall$ $t \in [0, \epsilon)$.

$\Rightarrow$ $\exists$ 1-parameter family of maps
$\varphi_t : M \to M$ which are generated by
$-W$

$$\partial_t \varphi_t(p) = -W(\varphi_t(p), t)$$

$$\varphi_0 = id_M .$$ [closedness of M is used]

Remark: Existence and uniqueness of sol^n to RF for non-compact manifolds as well as manifolds w/ boundary is also true. [cf. noncompact manifolds
Shi, Bing Long Chen – Xiping Zhu
manifolds w/ boundary
- Michael Anderson
– Panagiotis Gianniotis and other authors]

To relate the sol^n of the RDTF to the RF

$$\bar{g}(t) = \varphi_t^* g(t) \qquad t \in [0, \epsilon)$$

Claim:- $\bar{g}(t)$ is a sol^n to the R.F.

$$\partial_t \bar{g}(t) = -2\overline{Ric}(t)$$

$$\left[\partial_t\left(\varphi_t^* F(t)\right) = \varphi_t^*\left(\mathcal{L}_{X(t)}F(t) + \partial_t F(t)\right)\right]$$

$$\partial_t \bar{g}(t) = \partial_t\left(\varphi_t^* g(t)\right)$$

$$= \varphi_t^*\left(\mathcal{L}_{-W(t)} g(t) + \partial_t g(t)\right)$$

$$= \varphi_t^*\left(\mathcal{L}_{-W(t)} g(t) - 2\,Ric(g(t)) + \mathcal{L}_{W(t)} g(t)\right)$$

$$= \varphi_t^*\left(-2\,Ric(g(t))\right)$$

$$= -2\,Ric\left(\varphi_t^* g(t)\right)$$

$$= -2\,Ric\left(\bar{g}(t)\right)$$

$\therefore\ \partial_t \bar{g}(t) = -2Ric(\bar{g}(t))$

$\Rightarrow$ $\bar{g}(t)$ is a sol^n to the RF on $[0,\epsilon)$ — existence of sol^n to the RF. $\square$

Uniqueness is still left:-

$\mathcal{E} : C^{\infty}(M) \to \mathbb{R}_{\geq 0} \quad \mathcal{E}(u) = \int_M |\nabla u|^2 \, vol$

$\downarrow$ negative gradient flow

$$\frac{\partial u(t)}{\partial t} = \Delta u(t) \text{ - Harmonic map heat flow.}$$

Uniqueness requires some results from harmonic map heat flow.

Topic for the presentation.

Given (M^n, g_0) closed $\exists!$ sol^n to the RF on $[0, \epsilon)$.

$u(t)$, parabolic eq^n

$u(x, 0) \geq C_0$

$u(x, t) \geq C_0$.

* maximum principle for sol^n to the RF.

- maximum principle for scalar equations.
- max. principle for symmetric 2-tensors.
- max. principle for systems.

$$\partial_t Rm = \Delta Rm + \text{extra terms}.$$

Theorem 2 (Charaterization of the existence time)

Sol^n to the RF. will exist as long as $|Rm|$ is bounded.

Theorem 1 (apriori estimates, Derivative estimates, Shi-type estimates)

If $|Rm| < C$ then $|\nabla^k Rm| < C'$.

Theorem (Compactness thm for solutions)

Under certain conditions, a sequence of sol^n converge to a limit which is also a sol^n to the RF.

Gromov-Hausdorff convergence.