Lecture – Failure of parabolicity of the Ricci flow

Recall :–

$L : \Gamma(E) \to \Gamma(F)$ of order m

$$L(u) = \sum_{|\alpha| \le m} L_\alpha \partial^\alpha u \qquad L_\alpha \in \mathrm{Hom}(E, F)$$

Total symbol $\xi \in \Gamma(T^*M)$

$$\sigma[L](\xi) : E \to F$$
$$\sum_{|\alpha| \le m} L_\alpha \xi^\alpha$$

principal symbol

$$\hat{\sigma}[L](\xi) : E \to F$$
$$= \sum_{|\alpha| = m} L_\alpha \xi^\alpha$$

$u : M \times [0, t) \to E$ w/

$$\frac{\partial u}{\partial t} = L(u) \qquad L \text{ is some diff. oper.}$$
$$u(x, 0) = u_0(x)$$

for L non-linear, the linearization of L at u_0 in the direction $v = u'(0)$

$$D[L] : \Gamma(E) \to \Gamma(F) \text{ s.t.}$$

$$D[L](v) = \frac{d}{dt} L(u(t)) \Big|_{t=0}$$

$u(t)$ is a time-dependent section.

$$-2Rc : \Gamma(S^2_+ T^*M) \longrightarrow \Gamma(S^2 T^*M)$$

$$g \longmapsto -2Ric(g)$$

<u>RF.</u> $\partial_t g(t) = -2Rc(g(t))$

if $\dfrac{\partial g_{ij}}{\partial t} = h_{ij}$, variation formulas tell us that

$$\partial_t R_{jk} = \frac{1}{2} g^{pq} \left(\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp} \right)$$

$$\therefore \left(DRic_g\right)(h)_{jk} = \frac{\partial}{\partial t} Ric_g(g(t)) \Big|_{t=0}$$

$$= \frac{1}{2} g^{pq} \Big(\nabla_q \nabla_j h_{kp} + \nabla_q \nabla_k h_{jp} - \nabla_q \nabla_p h_{jk} - \nabla_j \nabla_k h_{qp} \Big)$$

Principal symbol

$$\hat{\sigma}[D\mathrm{Ric}_g](\xi) : S^2_+ T^*M \longrightarrow S^2 T^*M$$

$$\big(\hat{\sigma}[D\mathrm{Ric}_g](\xi)\big)(h)_{jk} = \frac{1}{2} g^{pq} \Big(\xi_q \xi_j h_{kp} + \xi_q \xi_k h_{jp} - \xi_q \xi_p h_{jk} - \xi_j \xi_k h_{qp} \Big)$$

$$\hat{\sigma}[D(-2\mathrm{Ric}_g)](\xi)(h)$$

$$= g^{pq} \Big(\xi_q \xi_p h_{jk} + \xi_j \xi_k h_{pq} - \xi_q \xi_j h_{kp} - \xi_q \xi_k h_{jp} \Big).$$

In order for $-2\mathrm{Ric}$ to be elliptic, $\exists\, c > 0$ s.t. $\forall h$

$$\langle \hat{\sigma}[-2D\mathrm{Ric}_g](\xi)(h), h \rangle \geq c\, |\xi|^2 |h|^2$$

LHS from above

$$= g^{pq}\left(\xi_q\xi_p h_{jk} + \xi_j\xi_k h_{pq} - \xi_q\xi_j h_{kp} - \xi_q\xi_k h_{jp}\right)\underbrace{h^{jk}}_{h_{mn}g^{mj}g^{nk}}$$

choose $h_{jk} = \xi_j\xi_k \quad (h = \xi\otimes\xi)$

LHS =

$$g^{pq}\left(\xi_q\xi_p\xi_j\xi_k + \xi_j\xi_k\xi_p\xi_q - \xi_q\xi_j\xi_k\xi_p - \xi_q\xi_k\xi_j\xi_p\right)h^{jk}$$

$$= 0$$

$\therefore$ $-2\,Ric$ as an operator is NOT elliptic and RF is NOT parabolic $\Leftrightarrow$ $-2\,Ric$ has non-zero kernel as an operator.

Claim :- The failure of parabolicity is only due to the diffeo. invariance of Ric.

$g \rightsquigarrow Rc(g)$

$$\nabla_i R_{ik} = \frac{1}{2} \nabla_k R = \frac{1}{2} \nabla_k \left(g^{pq} R_{pq} \right).$$

contracted 2^{nd} Bianchi identity is a consequence of the diffeo. invariance of Rc.

$\{\varphi_t\}$ diffeo. generated by X w/ $\varphi_0 = id_M$

$$\varphi_t^*(R(g)) = R(\varphi_t^* g)$$

linearizing this eqn.

$$DR(\mathcal{L}_X g) = \frac{d}{dt} R(\varphi_t^* g)\Big|_{t=0} = \frac{d}{dt} \left(\varphi_t^*(R(g))\right)\Big|_{t=0}$$

$$= \mathcal{L}_X R = \nabla_X R.$$

$$\boxed{DR(\mathcal{L}_X g) = \nabla_X R}$$

Recall, $DR_g(h) = -g^{ij} g^{kl} \left(\nabla_i \nabla_j h_{kl} - \nabla_i \nabla_k h_{jl} + R_{ik} h_{jl} \right)$

$\qquad \overset{\shortparallel}{\frac{\partial}{\partial t} R(g(t))\Big|_{t=0}}$

here $h = \mathcal{L}_X g$, i.e. $h_{ij} = \nabla_i X_j + \nabla_j X_i$

$$DR(\nabla_i X_j + \nabla_j X_i)$$

$$= -g^{ij} g^{kl} \Big(\nabla_i \nabla_j (\nabla_k X_l + \nabla_l X_k)$$

$$- \cancel{\nabla_i \nabla_k (\nabla_j X_l} + \nabla_l X_j)$$

$$+ R_{ik} (\nabla_j X_l + \nabla_l X_j) \Big)$$

Ricci identity

$$\nabla_i (\nabla_k \nabla_j X_l - R_{jklm} X_m)$$

$$= \cancel{\nabla_i \nabla_k \nabla_j X_l} - \nabla_i R_{jklm} X_m$$

$$- R_{jklm} \nabla_i X_m$$

commute derivatives to get rid of all ∇X terms

$$= 2 X_k \nabla_i R_{ik}$$

$$2 X_k \nabla_i R_{ik} = X_k \nabla_k R$$

is true for any v.f. X

$$\Rightarrow \quad \nabla_i R_{ik} = \frac{1}{2} \nabla_k R$$

– contracted 2nd Bianchi identity.

conversely, the once contracted 2nd Bianchi identity gives the invariance of Ric. In fact, consider the map

$\partial : \Gamma(T^*M) \longrightarrow S^2 T^*M$

$(\partial(X))_{ij} = \nabla_i X_j + \nabla_j X_i$. Then

$$\left[\left(\left(DRc_g\right)\circ \partial\right)(X)\right]_{jk} =$$

$$\frac{1}{2}\nabla_p\nabla_j\left(\nabla_k X_p + \nabla_p X_k\right) + \frac{1}{2}\nabla_p\nabla_k\left(\nabla_j X_p + \nabla_p X_j\right)$$
$$-\frac{1}{2}\Delta\left(\nabla_j X_k + \nabla_k X_j\right) - \nabla_j\nabla_k\left(\operatorname{div} X\right)$$

$$= \frac{1}{2}\nabla_p\nabla_j\nabla_k X_p + \frac{1}{2}\nabla_p\nabla_j\nabla_p X_k + \frac{1}{2}\nabla_p\nabla_k\nabla_j X_p + \frac{1}{2}\nabla_p\nabla_k\nabla_p X_j$$
$$-\frac{1}{2}\Delta\left(\nabla_j X_k + \nabla_k X_j\right) - \nabla_j\nabla_k\left(\operatorname{div} X\right)$$

commuting covariant derivatives for the highlighted terms give

$$= \frac{1}{2}\nabla_p\nabla_j\nabla_k X_p + \frac{1}{2}\nabla_p\nabla_k\nabla_j X_p + \frac{1}{2}\nabla_p\left(\nabla_p\nabla_j X_k - R_{jpkm}X_m\right)$$
$$+\frac{1}{2}\nabla_p\left(\nabla_p\nabla_k X_j - R_{kpjm}X_m\right) - \frac{1}{2}\Delta\left(\nabla_j X_k + \nabla_k X_j\right)$$
$$-\nabla_j\nabla_k\left(\operatorname{div} X\right)$$

$$= \underline{\tfrac{1}{2} \nabla_p \nabla_j \nabla_k X_p} + \tfrac{1}{2} \nabla_p \nabla_k \nabla_j X_p - \nabla_j \nabla_k (\operatorname{div} X)$$

$$- \tfrac{1}{2} \nabla_p R_{jpkm} X_m - \tfrac{1}{2} R_{jpkm} \nabla_p X_m$$

$$- \tfrac{1}{2} \nabla_p R_{kpjm} X_m - \tfrac{1}{2} R_{kpjm} \nabla_p X_m$$

The underlined term on using the Ricci identity becomes

$$= \tfrac{1}{2} \nabla_j \nabla_p (\nabla_k X_p) - \tfrac{1}{2} R_{pjks} \nabla_s X_p - \tfrac{1}{2} R_{pjps} \nabla_k X_s$$

$$= \tfrac{1}{2} \nabla_j (\nabla_k \nabla_p X_p - R_{pkpl} X_l) - \tfrac{1}{2} R_{pjks} \nabla_s X_p + \tfrac{1}{2} R_{js} \nabla_k X_s$$

$$= \tfrac{1}{2} \nabla_j \nabla_k (\operatorname{div} X) + \tfrac{1}{2} \nabla_j R_{kl} X_l + \tfrac{1}{2} R_{kl} \nabla_j X_l - \tfrac{1}{2} R_{pjks} \nabla_s X_p$$

$$+ \tfrac{1}{2} R_{js} \nabla_k X_s$$

and similarly by interchanging $k \longleftrightarrow j$ and simplifying, we get.

$$[D(\mathrm{Rc}_g) \circ \partial](X)_{jk} = \tfrac{1}{2} \nabla_j R_{kl} X_l + \tfrac{1}{2} R_{kl} \nabla_j X_l$$

$$- \tfrac{1}{2} R_{pjks} \nabla_s X_p + \tfrac{1}{2} R_{js} \nabla_k X_s$$

$$+ \tfrac{1}{2} \nabla_k R_{jl} X_l + \tfrac{1}{2} R_{jl} \nabla_k X_l - \tfrac{1}{2} R_{pkjs} \nabla_s X_p + \tfrac{1}{2} R_{ks} \nabla_k X_s$$

$$- \frac{1}{2} \underline{\nabla_p R_{jpkm} X_m} - \frac{1}{2} R_{jpkm} \nabla_p X_m$$

$$- \frac{1}{2} \underline{\nabla_p R_{kpjm} X_m} - \frac{1}{2} R_{kpjm} \nabla_p X_m$$

Use the once contracted 2^{nd} Bianchi identity for the underlined terms and collecting terms we'll get that

$$[D(R_{eg}) \circ \partial](X)_{jk} = (X_p \nabla_p R_{jk} + R_{jp} \nabla_k X_p + R_{kp} \nabla_j X_p)$$

$$= [\mathcal{L}_X (R_{eg})]_{jk}.$$

Thus, Once contracted 2^{nd} Bianchi identity $\Longrightarrow$ diffeo. invariance of the Ricci tensor.

I mentioned this in the lecture but above is a proof. Also note that I am using once contracted 2^{nd} Bianchi identity to prove diff. invariance of Ric. If you start w/ $[DR_g \circ \partial](X)$ and proceed as above and use twice contracted 2^{nd} Bianchi iden. then you'll get the diffeo. invariance of the scalar curvature, the converse of which was done in the lectures. Thus Bianchi iden. $\Longrightarrow$ diffeo. invariance.

Idea for any other geometric flow

The diffeomorphism invariance of tensors involved gives you new identities for the tensor.

Exer. :- $\varphi_t^*(Ric(g)) = Ric(\varphi_t^* g)$ gives

$$\nabla_i R_{ijmk} = \nabla_k R_{jm} - \nabla_m R_{jk} \quad (\text{Ass. } 1)$$

and $\varphi_t^*(Rm(g)) = Rm(\varphi_t^* g)$

gives

algebraic & diff. Bianchi identity.

(cf. Hilbert)

Jerry Kazdan)

We'll prove that $\ker(\hat{\sigma}[DRc])$ is n-dim in every $\frac{n(n+1)}{2}$-dim fiber of S^2T^*M.

Want to analyze $\hat{\sigma}[DRc]$

Consider $\partial : \Gamma(T^*M) \to S^2T^*M$

$$(\partial X)_{ij} = \nabla_i X_j + \nabla_j X_i = (\mathcal{L}_X g)_{ij}$$

$$\hat{\sigma}(\partial)(\xi)_{ij} = \xi_i X_j + \xi_j X_i$$

$$D(Rc_g) \circ \partial : \Gamma(T^*M) \longrightarrow S^2T^*M$$

$$\hat{\sigma}[D(Rc_g)\circ\partial] = \hat{\sigma}[D(Rc)] \circ \hat{\sigma}[\partial].$$

Claim :- $\hat{\sigma}[D(Rc)] \circ \hat{\sigma}[\partial]$ is the zero map.

Calculate

$$\hat{\sigma}[\partial] = \xi_i X_j + \xi_j X_i$$

$$\hat{\sigma}[D(Rc)] = \frac{1}{2} g^{pq}\Big(\xi_q \xi_j h_{kp} + \xi_q \xi_k h_{jp} - \xi_q \xi_p h_{jk} - \xi_j \xi_k h_{qp}\Big)$$

$$\hat{\sigma}[DRc] \circ \hat{\sigma}[\partial]$$

$$= \frac{1}{2} g^{pq}\Big(\xi_q \xi_j (\xi_k X_p + \xi_p X_k)$$

$$+\xi_q\xi_k\left(\xi_j X_p+\xi_p X_k\right)$$

$$-\xi_q\xi_p\left(\xi_j X_k+\xi_k X_j\right)$$

$$-\xi_j\xi_k\left(\xi_q X_p+\xi_p X_q\right)\Big)$$

$$=\frac{1}{2}g^{pq}\Big(\xi_q\xi_j\xi_k X_p+\xi_q\xi_j\xi_p X_k$$

$$+\xi_q\xi_k\xi_j X_p+\xi_q\xi_k\xi_p X_k$$

$$-\xi_q\xi_p\xi_j X_k-\xi_q\xi_p\xi_k X_j$$

$$-\xi_j\xi_k\xi_q X_p-\xi_j\xi_k\xi_p X_q\Big)$$

$$=0$$

$$\hat\sigma[DRc]\circ\hat\sigma[\partial]=0$$

$$\Longrightarrow\quad \underbrace{\mathrm{im}\left(\hat\sigma[\partial](\xi)\right)}_{n\text{-dim}}\subseteq \ker\left(\hat\sigma[DRc](\xi)\right)$$

$$\boxed{\dim\left(\ker\left(\hat\sigma[DRc](\xi)\right)\right)\geq n}\quad -(1)$$

__digression:-__

$D(Reg) \circ \delta$ is a priori a 3^{rd}-order diff. operator.

$$\therefore \quad Rc(\varphi^* g) = \underbrace{\varphi^*(Rc(g))}_{\underbrace{\mathcal{L}_X(Rc)}_{\text{order } 1}}$$

If $\hat{\sigma}[D[Ric] \circ \delta] = 0$.

We now show that

$$\dim(\ker(\hat{\sigma}[DRic]) \leq n$$

We introduce Bianchi operator

$$B_g : S^2 T^*M \longrightarrow \Gamma(T^*M)$$

$$B_g(h)_k = g^{ij}\left(\nabla_i h_{jk} - \frac{1}{2} \nabla_k h_{ij}\right)$$

$B_g(Ric) = 0$. (twice contracted 2^{nd} Bianchi identity).

$$\left(\hat{\sigma}[B_g](\xi)(h)\right)_k = g^{ij}\left(\xi_i h_{jk} - \tfrac{1}{2}\xi_k h_{ij}\right)$$

$$B_g \circ DRic : \Gamma(S^2_+ T^*M) \longrightarrow \Gamma(T^*M)$$

$$\hat{\sigma}(B_g \circ DRic) = \underbrace{\hat{\sigma}(B_g) \circ \hat{\sigma}[DRic]}_{\text{zero map.}}$$

$\underrightarrow{\text{Exercise.}}$ check by using the expression from $\hat{\sigma}[DRic]$ that indeed $\hat{\sigma}(B_g) \circ \hat{\sigma}[DRic] = 0$.

$$\operatorname{im}\left(\hat{\sigma}[DRic](\xi)\right) \subseteq \ker\left(\hat{\sigma}[B_g]\right) \longrightarrow (2)$$

Let $K_\xi = \ker\left(\hat{\sigma}[B_g](\xi)\right) \subseteq S^2 T^*M$

and

$$A_\xi = \left\{ \xi \otimes X + X \otimes \xi - \langle \xi, X \rangle g \;\middle|\; X \in \Gamma(T^*M) \right\} \subseteq S^2 T^*M$$

If $\xi \neq 0$, $\dim A_\xi = n$. as A_ξ has trivial kernel.

$$\langle \hat{\sigma}[Bg](\xi)(h), X\rangle$$
$$= \langle \xi_i h_{ij} - \tfrac{1}{2}\xi_j h_{ii}, X_j\rangle$$
$$= \tfrac{1}{2}\langle \xi\otimes X + X\otimes\xi - \langle\xi, X\rangle g, h\rangle$$

$\Rightarrow$ $A_\xi : \Gamma(T^*M) \to S^2(T^*M)$ is the adjoint of $\hat{\sigma}[Bg]$

and

$$\mathrm{Ker}(\hat{\sigma}[Bg]) = K_\xi = A_\xi^{\perp}$$

$$\Rightarrow \dim(\mathrm{Ker}(\hat{\sigma}[Bg](\xi))) = \frac{n(n+1)}{2} - n$$
$$= \frac{n(n-1)}{2}.$$

$$\therefore [\hat{\sigma}[DRc](\xi)(h)]_{jk} = \tfrac{1}{2} g^{pq}\{ \xi_q\xi_j h_{kp} + \xi_q\xi_k h_{jp} - \xi_p\xi_p h_{jk} - \xi_j\xi_k h_{pq}\}$$

If $h\in K_\xi$, i.e. h satisfies $\xi_i h_{ij} = \frac{1}{2}\xi_j h_{ii}$, we get

$$[\hat{\sigma}[DRc](\xi)(h)]_{jk} = \tfrac{1}{2}\{ \xi_q\xi_j h_{kq} + \xi_q\xi_k h_{jq} - |\xi|^2 h_{jk} - \xi_j\xi_k h_{qq}\}$$
$$= \tfrac{1}{2}\{\xi_j \tfrac{1}{2}\xi_k \,\mathrm{tr}h + \xi_k\tfrac{1}{2}\xi_j\,\mathrm{tr}h - |\xi|^2 h_{jk} - \xi_j\xi_k \,\mathrm{tr}h\}$$
$$= -\tfrac{1}{2}|\xi|^2 h_{jk}$$

$\therefore\ \hat{\sigma}[DRc_g](\xi) = -\frac{1}{2}|\xi|^2 \mathrm{id}_{K_\xi}$ and is an automorphism on each fiber.

$\Rightarrow\ \dim(\mathrm{im}\,\hat{\sigma}[DRc](\xi)) \geq \dim K_\xi = \frac{n(n-1)}{2}$

$\because\ \dim(\mathrm{im}\,\hat{\sigma}[DRc]) + \dim(\mathrm{Ker}(\hat{\sigma}[DRc]) = \frac{n(n+1)}{2}$

we get

$$\dim(\mathrm{Ker}(\hat{\sigma}[DRc](\xi)] \leq n \quad — (2)$$

from eq (1) and (2)

$$\dim(\mathrm{Ker}(\hat{\sigma}[DRc](\xi)] = n$$

Thus $\hat{\sigma}[DRc]$ is an isomorphism on $\mathrm{Ker}(\hat{\sigma}[B_g])$

$\Rightarrow$ the only obstruction to the parabolicity of the RF is $\mathrm{im}(\hat{\sigma}[B_g])$ $\Rightarrow$ the obstruction is only the Bianchi identity. But $\because$ the Bianchi identity $\Leftrightarrow$ diffeo. invariance $\Rightarrow$ the only obstruction to the parabolicity of RF is diffeo. invariance of Ric.