Differential operators and basics of the Ricci flow

We recall that the square of the covariant derivative is

$$\nabla^2 : \Gamma(T^p_q M) \longrightarrow \Gamma(T^{p+2}_q M)$$

w/

$$\nabla^2_\alpha (X, Y, Z_1, \dots, Z_p) = \nabla_X (\nabla\alpha)(Y, Z_1, \dots, Z_p)$$

$$= [\nabla_X(\nabla\alpha(Y)) - \nabla\alpha(\nabla_X Y)](Z_1, \dots, Z_p)$$

$$= \nabla_X(\nabla_Y \alpha)(Z_1, \dots, Z_p) - \nabla_{\nabla_X Y}\alpha(Z_1, \dots, Z_p)$$

so in short

$$\nabla^2_{X,Y}\alpha = \nabla_X \nabla_Y \alpha - \nabla_{\nabla_X Y}\alpha .$$

For ex. in local coordinates

$$\nabla_i R_{jk} = (\nabla Rc)(\partial_i, \partial_j, \partial_k)$$

$$= (\nabla_{\partial_i} Rc)(\partial_j, \partial_k)$$

$$= \frac{\partial}{\partial x^i} R_{jk} - \Gamma_{ij}^{l} R_{lk} - \Gamma_{ik}^{l} R_{jl}$$

Similarly

$$\nabla_i R_{jklm} = \partial_i R_{jklm} - \Gamma_{ij}^{p} R_{pklm} - \Gamma_{ik}^{p} R_{jplm} - \Gamma_{il}^{p} R_{jkpm} - \Gamma_{im}^{p} R_{jklp}$$

for a function f, one can show that

$$\nabla_i \nabla_j f = (\nabla \nabla f)(\partial_i, \partial_j)$$
$$= \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^{k} \frac{\partial f}{\partial x^k}.$$

* We denote the m-th order derivative of α by $\nabla^m \alpha = \underbrace{\nabla \cdots \nabla}_{m\text{-times}} \alpha$

and its components will be denoted by

$$\nabla_{j_1} \cdots \nabla_{j_m} \alpha_{i_1 \ldots i_p}.$$

Lie Derivative

For $X, Y \in \Gamma(TM)$

$$\mathcal{L}_X Y = [X, Y]$$ w/

$[X, Y] \in \Gamma(TM)$ s.t.

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

If θ is a 1-form then

$$(\mathcal{L}_X \theta)(Y) = X(\theta(Y)) - \theta([X, Y])$$

even though one doesn't need a metric to define the Lie derivative, it is related to the L-C connection via the formula

$$(\mathcal{L}_X A)(Y_1, \dots, Y_p, \theta_1, \dots, \theta_q)$$

$$= X(A(Y_1, \dots, Y_p, \theta_1, \dots, \theta_q))$$

$$- \sum_{1 \le i \le p} A(Y_1, \dots, [X, Y_i], \dots, Y_p, \theta_1, \dots, \theta_q)$$

$$- \sum_{1 \le j \le q} A(Y_1, \dots, Y_p, \theta_1, \dots, \mathcal{L}_X \theta_j, \dots, \theta_q)$$

In particular, when $A = g$ then

$$(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$$

whose expression in coordinates give

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i .$$

If $X = \nabla f$ for $f \in C^\infty(M)$ then

$$(\mathcal{L}_X g)_{ij} = (\mathcal{L}_{\nabla f} g)_{ij} = \nabla_i \nabla_j f + \nabla_j \nabla_i f$$
$$= 2 \nabla_i \nabla_j f.$$

Ricci Identities

We have

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_{k_1, \dots, k_r} = - \sum_{l=1}^{r} R_{ijk_l}{}^{m} \alpha_{k_1 \cdots k_{l-1} m k_{l+1} \cdots k_r}$$

So for a 1-form α,

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \alpha_k = - R_{ijkl} \alpha_l$$

or for a 2-tensor β

$$\nabla_i \nabla_j \beta_{kl} - \nabla_j \nabla_i \beta_{kl} = - R_{ijkm} \beta_{ml} - R_{ijlm} \beta_{km}.$$

The divergence of a $(p,0)$-tensor is

$$(\operatorname{div}(\alpha))_{i_1 \cdots i_{p-1}} = g^{jk} \nabla_j \alpha_{k i_1 \cdots i_{p-1}}$$

$$= \nabla_j \alpha^{j}{}_{i_1 \cdots i_{p-1}}$$

for 1-forms α, $\operatorname{div} \alpha$ is a function,

$\operatorname{div} \alpha = \nabla_i \alpha_j$.

Laplacian

Laplacian Δ on functions is div (grad)

i.e,

$$\Delta = \operatorname{div} \nabla = g^{ij} \nabla_i \nabla_j$$

$$= g^{ij} \left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right)$$

For tensors, again

$$\Delta = \operatorname{div}(\operatorname{grad}) = \operatorname{trace}_g \nabla^2$$

$$= g^{ij} \nabla_i \nabla_j$$

<u>Exercise</u>:- Prove the Bochner formula for $|\nabla f|^2$

i.e, $\forall f \in C^\infty(M)$

$$\Delta |\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla_i f \nabla_j f + 2\nabla_i f \nabla_i(\Delta f).$$

Conclude that if $Rc \geq 0$, $\Delta f \equiv 0$ and $|\nabla f| = 1$ then ∇f is parallel.

We also have the divergence theorem and integration by parts formula. Let M^n be a closed manifold and $u, v \in C^\infty(M)$ and $X \in \Gamma(TM)$. Then

$$\int_M \operatorname{div} X \operatorname{vol} = 0 \quad \text{and so}$$

$$\int_M \Delta u \operatorname{vol} = 0$$

$$\int_M u \Delta v \operatorname{vol} = \int_M v \Delta u \operatorname{vol} \quad \text{(Integration by parts).}$$

Ricci Flow

Let (M^n, g_0) be given.

Def^n A Ricci flow on (M^n, g_0) is a family of metrics $(g(t))_{t\in[0,\epsilon)}$ s.t.

$$\partial_t g(t) = -2\mathrm{Ric}(g(t))$$
$$g(0) = g_0 .$$

RF

ϵ depends on M^n and g_0.

Examples :-

① If g_0 is Ricci-flat, i.e, $\mathrm{Ric}=0 \Rightarrow g(t)=g_0$ $\forall\, t$ is a sol^n. Note that $t\in(-\infty,\infty)$ in this case. e.g. when $M=\mathbb{R}^n$ or flat torus.

② Let g_0 be an Einstein metric, i.e,

$$\mathrm{Ric}(g_0) = \lambda g_0 \text{ for some } \lambda\in\mathbb{R}.$$

Then $g(t) = (1-2\lambda t) g_0$ is a sol^n to RF

as $\partial_t g(t) = -2\lambda g_0 = -2 Ric(g_0)$

$$= -2 Ric\left((1-2\lambda t) g_0\right)$$
$$= -2 Ric(g(t)).$$

$\therefore$ $g(t) = 0$ at $t = \dfrac{1}{2\lambda}$.

If $\lambda > 0$ then the solutions are shrinking as $g(t)$ is shrinking from g_0.

$\lambda = 0$ static sol^n

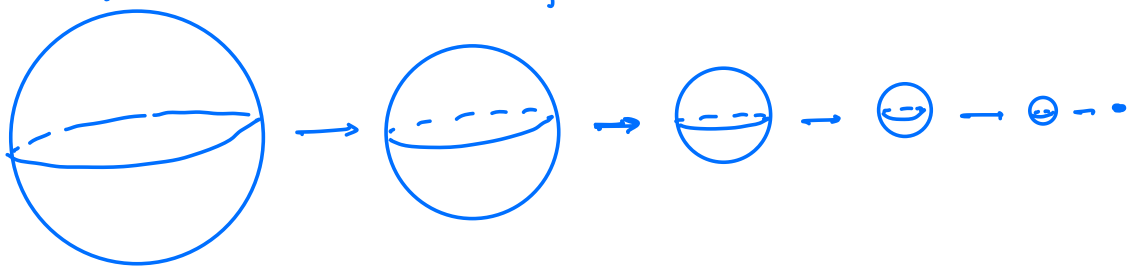
$\lambda < 0$ expanding sol^n.

A concrete case is that of (S^n, g_0). Here

$$Ric(g_0) = (n-1) g_0 \implies g(t) = (1-2(n-1)t) g_0$$

is a sol^n to RF. and the sol^n exists till $T = \dfrac{1}{2(n-1)}$.

Pictorially, the flow runs by shrinking the sphere until it becomes a point.



Def^n:- • $g(t)$ is called an eternal sol^n if $t \in (-\infty, \infty)$.

• $g(t)$ is an ancient sol^n if $t \in (-\infty, c)$, $c < \infty$.

(e.g. round sphere)

• $g(t)$ is an immortal sol^n if $t \in (\alpha, \infty)$, $\alpha > -\infty$.

Symmetries of RF

* $(M, g(t))_{t \in I}$ is RF $\longrightarrow$ $(M, g_{t-t_0})_{t \in I + t_0}$ is a RF.

* Parabolically rescaling a RF gives another RF
↳ "time scales like $(\text{distance})^2$"

i.e. if $g(t)$ is a RF then

$$\hat{g}(x,t) = \lambda g\left(x, \frac{t}{\lambda}\right), \quad t \in [0, \lambda T] \text{ is also a RF.}$$

as $\partial_t \hat{g}(x,t) = \lambda \cdot \frac{1}{\lambda} \partial_t g\left(x, \frac{t}{\lambda}\right) = -2\,\mathrm{Ric}(g(x, t/\lambda))$
$= -2\,\mathrm{Ric}(\hat{g})$.

* Diffeomorphism invariance: If $\varphi: M \to M$ is a diffeo. and $g(t)$ is a RF then so is $\varphi^* g(t)$.

Ricci flow regarded as a heat eqⁿ

Defⁿ:- Local coordinates (x^i) are called harmonic if $\Delta x^i = 0$.

$$\therefore\ 0 = \Delta x^i = g^{jk}\left(\partial_j \partial_k - \Gamma^{\ell}_{jk}\partial_\ell\right)x^i = -g^{jk}\Gamma^{i}_{jk}.$$

Lemma:- For $p \in M$ $\exists$ harmonic coordinates in some nbd. of p.

Lemma:- In harmonic coordinates,

$$-2R_{ij} = \Delta(g_{ij}) + Q_{ij}(g^{-1}, \partial g)$$

Laplacian of components

a term involving quadratic expressions in metric inverse g^{-1} and ∂g.

* **Notation**:- For tensors A, B

$A * B$ would mean "some linear combination of traces of $A \otimes B$ w/ coefficients that do not depend on A or B".

So various contractions b/w tensors whose precise forms are not important.

$\therefore$ in harmonic coordinates, indeed

$$\partial_t g_{ij} = \Delta(g_{ij}) + Q_{ij}(g^{-1}, \partial g).$$

<u>Proof of the lemma</u> :-

Recall the formula for R_{jk} in local coordinates give

$$-2R_{jk} = -2\left(\partial_q \Gamma^q_{jk} - \partial_j \Gamma^q_{qk} + \Gamma^p_{jk}\Gamma^q_{qp} - \Gamma^p_{qk}\Gamma^q_{jp}\right)$$

$$= -2\Big(\partial_q\Big(\tfrac{1}{2} g^{qr}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk})\Big) - \partial_j\Big(\tfrac{1}{2} g^{qr}(\partial_q g_{kr} + \partial_k g_{qr} - \partial_r g_{kq})\Big) + \Gamma^p_{jk}\Gamma^q_{qp} - \Gamma^p_{qk}\Gamma^q_{jp}\Big)$$

$$= -\partial_q\big(g^{qr}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk})\big) + \partial_j\big(g^{qr}(\partial_q g_{kr} + \partial_k g_{qr} - \partial_r g_{kq})\big) + g^{-1} * g^{-1} * \partial g * \partial g$$

(we used the coordinate expression for Γ^k_{ij})

$$= g^{qr}\left(-\underline{\partial_q\partial_j g_{rk}} - \partial_q\partial_k g_{rj} + \underline{\partial_q\partial_r g_{jk}}\right)$$

$$+ g^{qr}\left(\underline{\partial_j\partial_q g_{kr}} + \partial_j\partial_k g_{qr} - \partial_j\partial_r g_{kq}\right)$$

$$+ \; g^{-1} * g^{-1} * \partial g * \partial g \longrightarrow (*)$$

— terms cancel

— is the $\Delta(g_{jk})$ term on contracting q and r

The remaining 3 terms can be written in terms of partial derivatives of $\Gamma * g$. e.g.

$$g^{qr}\left(-\partial_q\partial_k g_{rj} + \tfrac{1}{2}\partial_j\partial_k g_{qr}\right) = -g^{qr}\partial_k\left(\Gamma^s_{qr} g_{sj}\right)$$

as

$$-g^{qr}\partial_k\left(\Gamma^s_{qr} g_{sj}\right)$$

$$= -g^{qr}\partial_k\left(\tfrac{1}{2} g^{sl}\left(\partial_q g_{rl} + \partial_r g_{ql} - \partial_l g_{qr}\right) g_{sj}\right)$$

$$= -g^{qr}\partial_k\left(\tfrac{1}{2}\left(\partial_q g_{rj} + \partial_r g_{qj} - \partial_j g_{qr}\right)\right)$$

$$= -g^{qr}\left(\tfrac{1}{2}\left(\partial_k\partial_q g_{rj} + \partial_k\partial_r g_{qj} - \partial_k\partial_j g_{qr}\right)\right)$$

$$= -\partial_k\partial_q g_{rj} + \tfrac{1}{2}\partial_k\partial_j g_{qr}$$

and the remaining term $g^{qr}\left(-\partial_j\partial_r g_{kq} + \frac{1}{2}\partial_k\partial_j g_{qr}\right)$ in (*) $= -g^{qr}\partial_j(\Gamma^s_{qr} g_{sk})$

So (*) becomes

$$-2R_{jk} = \Delta(g_{jk}) \underline{- g^{qr}\partial_k(\Gamma^s_{qr} g_{sj}) - g^{qr}\partial_j(\Gamma^s_{qr} g_{sk})}$$

$$+ g^{-1} * g^{-1} * \partial g * \partial g$$

The underlined ______ terms are zero in harmonic coordinates and so

$$-2R_{jk} = \Delta(g_{jk}) + \underbrace{g^{-1} * g^{-1} * \partial g * \partial g}_{Q_{ij}(g^{-1}, \partial g)}$$

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