$V = \mathrm{End}_{SA}(W) \underset{\text{Subbundle}}{\leq} W^* \otimes W$

$u \in V$ and $\omega \in W$ is an eigensection w/ eigenvalue $\lambda \in \mathbb{R}$, i.e., $u(\omega) = \lambda\omega$. Then the lifts satisfy

$$\tilde{u}(\tilde{\omega}) = \lambda\tilde{\omega}.$$

So for $r = \mathrm{rank}(W)$, and $u \in V_p$ given, we let $\lambda_1(u) \geq \lambda_2(u) \geq \ldots \geq \lambda_r(u)$ be ordered eigenvalues of u. Let

$$\Gamma = \{(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}^r \mid \lambda_1 \geq \cdots \geq \lambda_r\}$$

So we have the following criteria for invariance under parallel translations:-

Let $G : \Gamma \to \mathbb{R}$ be a function. For $c \in \mathbb{R}$, define

$$\mathcal{K} = \{u \in \mathcal{V} \mid G(\lambda_1(u), \ldots, \lambda_r(u)) \leq c\}.$$

Then $K \subset \mathcal{V}$ is invariant under parallel translation.

Also, recall that the ODE corresponding to the evolution of Rm is $\frac{d}{dt} M = M^2 + M^{\#}$

which for the eigenvalues

$\lambda(t) \geq \mu(t) \geq \nu(t)$ satisfy

$$\frac{d}{dt}\lambda = \lambda^2 + \mu\nu, \quad \frac{d}{dt}\mu = \mu^2 + \lambda\nu, \quad \frac{d}{dt}\nu = \nu^2 + \lambda\mu.$$

Now we can come up w/ many sets $\mathcal{K}$ which are invariant under parallel translations, closed and convex and which are preserved by the ODE.

① Let $C_0 \in \mathbb{R}$ and let

$$\mathcal{K} = \{ M \mid \lambda(M) + \mu(M) + \nu(M) \geq C_0 \}$$

$\therefore$ trace is a linear function $\Rightarrow$ convex $\Rightarrow$ K is a closed and convex function.

Let's see if $\mathcal{K}$ is preserved by the ODE or not.

$$\frac{d}{dt}(\lambda + \mu + \nu) = \frac{1}{2}\left[(\lambda+\mu)^2 + (\mu+\nu)^2 + (\lambda+\nu)^2\right]$$

$= \frac{2}{3}(\lambda_1+\lambda_2+\lambda_3) \geq 0$

$\therefore$ $\mathcal{K}$ is indeed preserved by K

$\Rightarrow$ if a curvature-like tensor is in $\mathcal{K}$ at $t=0$ then it remains so $\forall t$.

In $n=3$, the condition is precisely $R(0) \geq C_0 \Rightarrow R(t) \geq C_0$ which we already knew. (lower bound of R is preserved)

② $\mathcal{K} = \{ M \mid \nu(M) \geq 0 \}$ so the lowest eigenvalue ≥ 0.

note that $\nu : V_x \to \mathbb{R}$ is a concave function $\Rightarrow$

$$\nu(sM_1 + (1-s)M_2) \geq s\nu(M_1) + (1-s)\nu(M_2)$$

$$\geq 0$$

$\therefore$ $sM_1 + (1-s)M_2 \in \mathcal{K} \Rightarrow$ convex.

now

$$\frac{d\nu}{dt} = \nu^2 + \lambda\mu \geq 0 \quad \text{whenever } \nu \geq 0$$

In fact if $\nu_0 > 0$ then $\frac{d\nu}{dt} > 0 \Rightarrow$ $\nu(t) > 0$.

If $\nu(0) = 0$ and $\mu(0) > 0$ then still same thing holds.

If $\nu(0) = 0$ and $\mu(0) = 0$ and $\lambda(0) = 0$ then they all remain 0. If $\lambda(0) > 0$ then $\nu(t) = \mu(t) = 0$ and $\lambda(t) > 0$.

In any case K is preserved by the ODE.

$\therefore$ By the max. principle, $Rm \geq 0$ is preserved along the Ricci flow.

③ Let $K = \{ M \mid \mu(M) + \nu(M) \geq 0 \}$
show that K is closed, convex and is preserved by the ODE.

conclusion:- $Rc \geq 0$ is preserved already knew.

④ (Ricci pinching is preserved)

If $\lambda(Rm) \leq C(\mu(Rm) + \nu(Rm))$ for $C \geq 1/2$ then it remains so.

$$K = \{ M \mid \lambda(M) \leq C(\mu(M) + \nu(M)) \}$$

for a given $C \geq 1/2$.

if $C = 1/2$ then $\because \lambda(M) \geq \frac{1}{2}(\mu + \nu)$

and $\lambda(M) \leq \frac{1}{2}(\mu + \nu)$

we, get $\lambda = \frac{1}{2}(\mu + \nu)$

Also $\lambda \geq \mu \Rightarrow \frac{\lambda+\nu}{2} \geq \frac{\mu+\nu}{2}$

$\Rightarrow \frac{\lambda+\nu}{2} \geq \lambda \Rightarrow \frac{\nu}{2} \geq \frac{\lambda}{2}$

$\Rightarrow \lambda = \nu = \mu.$

$\therefore \lambda(t) = \nu(t) = \mu(t) \ \forall t \Rightarrow$

$\mathcal{K}$ is preserved by the ODE.

So let $C > \frac{1}{2}$ and $\lambda(0) \geq \mu(0) \geq \nu(0)$

then we have $\mu(0) + \nu(0) \geq 0$.

Because we already have $\lambda(0) \geq \frac{1}{2}(\mu(0)+\nu(0))$

So we can never have $\lambda(0) \leq C(\mu(0)+\nu(0))$

w/ $C > 1/2$

if $\mu(0)+\nu(0) < 0$.

now $\frac{d}{dt}(\mu+\nu) = \mu^2+\nu^2+\lambda(\mu+\nu)$

So either i) $\nu(0) = \mu(0) = \lambda(0) = 0$

then $\nu(t) = \mu(t) = \lambda(t) = 0$

or

ii) $\mu(0) + \nu(0) > 0 \Rightarrow \frac{d}{dt}(\mu+\nu) > 0$

$\Rightarrow (\mu+\nu)(t) > 0 \quad \forall\, t > 0.$

case i) trivially satisfies the condition that $\mathcal{K}$ is preserved by the ODE.

so, we only look at case ii).

so we can take log

$$\frac{d}{dt}\log\left(\frac{\lambda}{\mu+\nu}\right) = \frac{1}{\lambda(\nu+\mu)}\left((\nu+\mu)\frac{d\lambda}{dt} - \lambda\frac{d}{dt}(\nu+\mu)\right)$$

$$= \frac{1}{\lambda(\mu+\nu)}\left((\nu+\mu)(\lambda^2+\mu\nu) - \lambda(\nu^2+\lambda\mu+\mu^2+\lambda\nu)\right)$$

$$= \frac{\mu^2(\nu-\lambda) + \nu^2(\mu-\lambda)}{\lambda(\mu+\nu)} \leq 0$$

$\lambda(\mu+\nu)$

$$\therefore \quad \frac{\lambda(t)}{\mu(t)+\nu(t)} \leq \frac{\lambda(0)}{\mu(0)+\nu(0)} \leq C$$

Hence K is preserved by the ODE.

$\therefore$ if the initial eigenvalues of Rm have the pinching estimate then it is preserved.

Note that we have

$$C\, Rc \geq \lambda(Rm) g \geq \frac{1}{3} R g$$

So the above estimate in other words say that along the R.F.

$$Rc \geq \epsilon R g \text{ is preserved.}$$

for some $\epsilon \leq \breve{1}/3$. which we have already proved.

(5) (Ricci pinching improves).

Let $C_0 > 0$, $C_1 \geq \frac{1}{2}$, $C_2 < \infty$ and $\delta > 0$.
$(\delta < 1)$

and let

$$\mathcal{K} = \left\{ M \;\middle|\; \begin{array}{l} \nu + \mu + \lambda \geq C_0 \\ \lambda \leq C_1(\mu+\nu), \\ \lambda - \nu - C_2(\lambda+\mu+\nu)^{1-\delta} \leq 0 \end{array} \right\}$$

$\lambda - \nu - C_2(\lambda+\mu+\nu)^{1-\delta}$ is convex.

If $M \in \mathcal{K}$ then $\mu(M) + \nu(M) > 0$ by the first two inequalities.

(if $\mu + \nu \leq 0 \implies C_1(\mu+\nu) \leq 0$

$\Rightarrow \quad \lambda \leq 0$

But then $\quad \nu + \lambda + \mu \not\leq c_0)$

The first two are already preserved along RF.

We compute

$$\frac{d}{dt} \log \left(\frac{\lambda - \nu}{(\lambda + \mu + \nu)^{1-\delta}} \right)$$

(exercise)

$$= \delta(\nu + \lambda - \mu) - \frac{(1-\delta)\left[(\nu+\mu)\mu + (\mu - \nu)\lambda + \mu^2\right]}{(\lambda + \mu + 2\nu)}$$

$$\leq \delta(\nu + \lambda - \mu) - \frac{(1-\delta)\mu^2}{\lambda + \mu + 2\nu}$$

(note that $\nu + \lambda - \mu \leq \lambda \leq c_1(\mu + \nu) \leq 2c_1\mu$

$\Rightarrow \mu > 0$).

$\Rightarrow (\mu+\nu)\mu + (\mu - \nu)\lambda > 0$ and $(1-\delta) > 0 \Rightarrow$ the term we are neglecting is negative)

So note that $\mu+\lambda \le 2\lambda \le 2C_1(\mu+\nu)$

$$\frac{\mu^2}{\lambda+\mu+\nu} = \frac{\mu\cdot\mu}{\lambda+\mu+\nu} \qquad \left(\mu \ge \frac{\mu+\nu}{2}\right) \quad \frac{1}{\lambda+\mu+\nu} \ge \frac{1}{3\lambda} \qquad \ge \frac{\mu(\mu+\nu)}{6\lambda} \ge \frac{1}{6C_1}\mu$$

and $\nu+\lambda-\mu \le \lambda \le C_1(\mu+\nu) \le 2C_1\mu$

$$\therefore \frac{d}{dt}\log\left(\frac{\lambda-\mu}{(\lambda+\mu-\nu)^{1-\delta}}\right) \le \frac{\delta}{1-\delta}(2C_1\mu) - \frac{1}{6C_1}\mu$$

so if $\dfrac{\delta}{1-\delta} \le \dfrac{1}{12C_1^2} \Rightarrow \dfrac{2C_1\mu\delta}{1-\delta} \le \dfrac{1}{6C_1}\mu$

and we'll be done.

$$\therefore \quad \lambda-\nu \le C_2(\lambda+\mu+\nu)^{1-\delta}$$

<u>Exercise</u> Prove that this is equivalent to

$$\frac{\left|Rc - \frac{1}{3}Rg\right|}{R} \le CR^{-\delta}.$$

$\square$