Summary

$u : M^n \times [0,T) \longrightarrow \mathbb{R}$

$$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle$$

$g(t)$ is a family of metrics, $X(t)$ is a family of v.f.s.

If $u(x,0) \geq C$ $\forall\, x \in M$ then
$u(x,t) \geq C$ $\forall\, x \in M,\ t \in [0,T)$.

- $$\frac{\partial u}{\partial t} \geq \Delta_{g(t)} u + \langle X, \nabla u \rangle + \beta u$$
$\beta : M \times [0,T) \longrightarrow \mathbb{R}$

If $\forall\, \tau \in [0,T)$ $\exists\, C_\tau < \infty$ s.t. $\beta(x,t) \leq C_\tau$ $\forall\, x \in M$ and $t \in [0,\tau]$.
If $u(x,0) \geq C$ $\forall\, x \in M$ then $u(x,t) \geq C$ $\forall\, x \in M,\ t \in [0,T)$.

- $$\frac{\partial u}{\partial t} \geq \Delta u + \langle x, \nabla u \rangle + F(u)$$

$F : \mathbb{R} \to \mathbb{R}$, locally Lipschitz.

Suppose $\exists$ C s.t. $u(x,0) \geq C$ $\forall x \in M$.
Let φ be a solution to the ODE

$$\frac{d\varphi}{dt} = F(\varphi)$$
$$\varphi(0) = C \quad .$$

Then $u(x,t) \geq \varphi(t)$ $\forall x \in M, t \in [0,T)$.

Maximum principle for tensors

R - scalar function
$|Ric|^2$, $|Rm|^2$

- max. principle for symmetric 2-tensors
  " Hamilton '82 - 3-manifolds ... "
- max. principle for an arbitrary section of a vector bundle - Hamilton '86
  " Four manifolds w/ positive curvature operator".

$A \geq 0 \quad (A > 0)$

Theorem :- Let M^n be a closed manifold w/ a family of metrics $g(t)$. Let $\alpha(t)$ be a family of symmetric 2-tensors on M.

$$\frac{\partial \alpha}{\partial t} \geq \Delta_{g(t)} \alpha + \beta$$

where $\beta(\alpha, g, t)$ is a symmetric 2-tensor which is locally Lipschitz in all of its arguments and it satisfies the null eigenvector assumption :- If $V(x,t)$ is a null eigenvector of α, i.e., $(\alpha_{ij} V^i)(x,t) = 0$ then $\beta(V,V)(x,t) = (\beta_{ij} V^i V^j)(x,t) \geq 0$.

$\alpha_{ij} V^i V^j = 0$

If $\alpha(0) \geq 0$

then $\alpha(t) \geq 0$ s.t. $\alpha(t)$ exists.

Proof :-

Sketch :- $\alpha > 0 \quad \forall \quad 0 \le t < t_0$

but at (x_0, t_0) $\exists$ $v \in T_{x_0} M^n$ s.t.

$$(\alpha_{ij} v^j)(x_0, t_0) = 0.$$

Then $(\alpha_{ij} w^i w^j)(x, t) \ge 0 \quad \forall\, x \in M$

and $t \in [0, t_0]$. and $w \in T_x M^n$.

$$\frac{\partial}{\partial t}(\alpha_{ij} v^i w^j) = \left(\frac{\partial}{\partial t}\alpha_{ij}\right) v^i w^j + \alpha_{ij}\left(\frac{\partial}{\partial t} v^i\right) w^j + \alpha_{ij} v^i (\partial_t w^j)$$

$$\Delta(\alpha_{ij} v^i w^j) = 3 \text{ terms}$$

Extend the vector v in a space-time nbd of (x_0, t_0) s.t. $V(x_0, t_0) = v$ and

$$\left.\begin{aligned} \frac{\partial V}{\partial t}(x_0, t_0) &= 0 \\ \nabla V(x_0, t_0) &= 0 \\ \Delta V(x_0, t_0) &= 0 \end{aligned}\right\} \longrightarrow (*)$$

To get the last two points in (*) we parallel translate v in space along the geodesic starting from x_0 in M $g(t_0)$ and take v independent of time.

$$\Delta v(x_0,t_0) = \sum_{i=1}^{n} \left[\nabla_{e_i}(\nabla_{e_i} v) - \nabla_{\nabla_{e_i} e_i} v \right]_{(x_0,t_0)}$$

$\{e_i\}$ o.n.b of $T_{x_0}M$

$$= \sum \left[\nabla_{e_i} 0 - \nabla_0 v \right] = 0.$$

you start w/ the v which is the null eigenvector and parallel translate it and take independent of time to get (*).

Then in any space-time nbd (x_0, t_0)

$$\frac{\partial}{\partial t}(\alpha_{ij} v^i v^j) = \left(\frac{\partial}{\partial t}\alpha_{ij}\right) v^i v^j$$

$$\geq (\Delta \alpha_{ij} + \beta_{ij}) v^i v^j$$

$$(\alpha_{ij} v^i v^j)(x_0, t_0) = 0$$

$$(\alpha_{ij} v^i v^j)(x, t_0) \geq 0 \quad \forall\, x \text{ in a nbd of } x_0.$$

$\Rightarrow \quad \Delta(\alpha_{ij} v^i v^j) \geq 0$

$\Rightarrow \quad (\Delta \alpha_{ij}) v^i v^j \geq 0$

$\beta_{ij} v^i v^j_{(x_0, t_0)} \geq 0$ (null-eigenvector assumption)

$\therefore \quad \frac{\partial}{\partial t}(\alpha_{ij} v^i v^j) \geq \Delta(\alpha_{ij} v^i v^j) + \beta_{ij} v^i v^j$

$\quad\quad \overset{\shortparallel}{(\partial_t \alpha_{ij}) v^i v^j} \qquad\qquad \geq 0$

at (x_0, t_0).

$\therefore$ if $\alpha_{ij} v^i v^j$ ever becomes zero then it cannot decrease further.

$\Rightarrow \quad \alpha(t) \geq 0 \qquad \forall \; x \in M$
$\qquad\qquad\qquad\qquad t \in [0, T)$.

☺

Proof of the theorem

Let $\tau \in (0, T)$. We'll show $\exists \; \delta \in (0, \tau]$ s.t. $\forall \; t_0 \in [0, \tau - \delta]$, if $\alpha \geq 0$ at t_0

then $\alpha \geq 0$ on $M \times [t_0, t_0+\delta]$.

fix any $t_0 \in [0, T-\delta]$. For any $0 < \epsilon \leq 1$

$$A_\epsilon(x,t) = \alpha(x,t) + \epsilon[\delta + (t-t_0)] \cdot g(x,t)$$

$x \in M$ and $t \in [t_0, t_0+\delta]$.

$$A_\epsilon(x,t_0) = \alpha(x,t_0) + \epsilon[\delta] g(x,t_0) > 0$$

and the term $\epsilon(t-t_0)g$ will make

$$\frac{\partial A_\epsilon}{\partial t} > \frac{\partial \alpha}{\partial t} \quad \text{for } t \in [t_0, t_0+\delta]$$

if we choose δ sufficiently small.

$$\frac{\partial A_\epsilon}{\partial t} = \frac{\partial \alpha}{\partial t} + \epsilon g + \epsilon[\delta + (t-t_0)]\frac{\partial g}{\partial t}$$

$$\therefore \Delta A_\epsilon = \Delta \alpha$$

$$\Rightarrow \frac{\partial A_\epsilon}{\partial t} \geq \Delta A_\epsilon + \beta + \epsilon g + \epsilon[\delta + (t-t_0)]\frac{\partial g}{\partial t}$$

$\Rightarrow$

$$\frac{\partial A_\epsilon}{\partial t} \geq \Delta A_\epsilon + \beta(A_\epsilon, g, t) + [\beta(\alpha, g, t) - \beta(A_\epsilon, g, t)]$$

$$+ \epsilon g + \epsilon [\delta + (t - t_0)] \frac{\partial g}{\partial t} .$$

$$\text{———} \quad (1)$$

We first choose $\delta_0 > 0$ depending on $g(t)$, $t \in [0, T]$ to be small enough so that on $M \times [t_0, t_0 + \delta_0]$,

$$\frac{\partial g}{\partial t} \geq -\frac{1}{4\delta_0} g$$

$\Rightarrow$

$$\epsilon g + \epsilon [\delta_0 + (t - t_0)] \frac{\partial g}{\partial t} \geq \frac{1}{2} \epsilon g$$

on $M \times [t_0, t_0 + \delta_0]$

Now, β is locally Lipschitz $\Rightarrow \exists$ K depends on α, g, t but not on ϵ

$$\beta(\alpha, g, t) - \beta(A_\epsilon, g, t) \geq -K\epsilon[\delta_0 + (t-t_0)g]$$
$$\geq -2K\epsilon\delta_0 g.$$

on $M \times [t_0, t_0+\delta_0]$.

We choose $\delta \in (0, \delta_0)$ small enough so that

$$\delta < \frac{1}{4K}$$

$$\Rightarrow \beta(\alpha, g, t) - \beta(A_\epsilon, g, t) > -\frac{1}{2}\epsilon g$$

$\therefore$ eqn ① with all the estimates give

$$\frac{\partial A_\epsilon}{\partial t} > \Delta A_\epsilon + \beta(A_\epsilon, g, t). \text{ on}$$
$$M \times [t_0, t_0+\delta]$$

Enough to prove that $A_\epsilon > 0$ on $M \times [t_0, t_0+\delta]$

Suppose not, i.e., $\exists$ $(x_1, t_1) \in M \times (t_0, t_0+\delta]$ $v \in T_{x_1}M^n$ s.t. $A_\epsilon > 0$ for all time $t_0 \leq t < t_1$

but $(A_\epsilon)_{ij} v^i (x_1, t_1) = 0$.

We extend this v.f. v to V as per $(*)$

$$0 \geq \frac{\partial}{\partial t}\left((A_\epsilon)_{ij} V^i V^j\right) = \left(\frac{\partial}{\partial t} A_\epsilon\right)_{ij} V^i V^j$$

$$> \left(\Delta(A_\epsilon V^i V^j\right) + \beta_{ij} V^i V^j \geq 0$$

contraction.

$\Rightarrow$ $A_\epsilon > 0$ on $M \times [t_0, t_0+\delta]$

$\therefore$ $\delta > 0$ depends only on $\left|\frac{\partial}{\partial t} g\right|$ and K and not on ϵ, we can let $\epsilon \searrow 0$

(Lipschitz of β)

$\Rightarrow$ $\alpha_{ij} \geq 0$ on $M \times [t_0, t_0+\delta]$.

$\boxtimes$

Topic for presentation :- Statement and proof of the v.b. version of the maximum principle.