M^n closed, g_0 $\exists!$ to the RF $(g(t))_{t\in[0,T]}$ w/ $g(0) = g_0$.

$\exists$ maximal time T s.t sol^n w/ initial cond. g_0, exists. $\nexists$ for time $T+\epsilon$.

Maximum principles

general idea :- parabolic or "heat-type" PDE will "average out" the values.

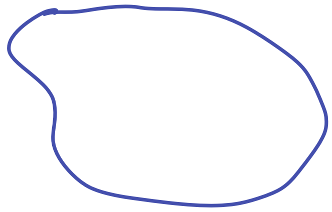
Heat eq^n. $\partial_t u = \dfrac{\partial^2 u}{\partial x^2}$, $u(x,t)$ is the temp. at x in time time t.

max principle :- temp. at the hottest point is non-increasing function of time.

" ——— " coldes point is non-decreasing funct. of time.

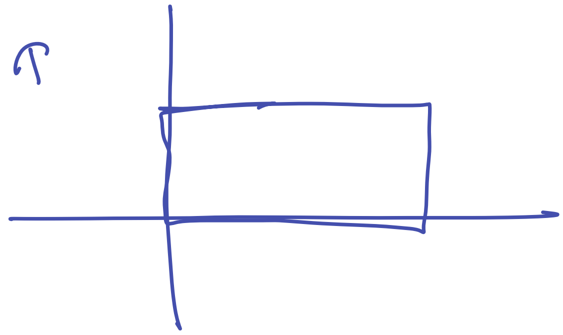
at the point of max. temp. u has a local maxima, $\Rightarrow$ $\dfrac{\partial^2 u}{\partial x^2} \leq 0 \Rightarrow \dfrac{\partial u}{\partial t} \leq 0$.

$\Rightarrow$ u is non-increasing.

U 

$U \times T$

Strong maximal principle



Weak max. principle for scalars

Prop^n 1 M^n closed

$u : M \times [0,T] \longrightarrow \mathbb{R}$ which is a sol^n to the heat eq^n

$$\frac{\partial u}{\partial t} = \Delta_g u$$

If $\exists$ $C_1, C_2 \in \mathbb{R}$ s.t. $C_1 \leq u(x,0) \leq C_2$ $\forall$ $x \in M^n$ then $C_1 \leq u(x,t) \leq C_2$ $\forall$ $x \in M^n$ and $t \in [0,T)$.

Def^n:- Let $g(t)$, $t \in [0,T)$ be a family of metrics, $X(t)$ be a family of v.f. on M^n.

$$\frac{\partial v}{\partial t} = \Delta_{g(t)} v + \langle X, \nabla v \rangle \quad \text{--- (1)}$$

We say that $u: M \times [0,T) \to \mathbb{R}$ is a supersolution to (1) at (x,t) if

$$\frac{\partial u}{\partial t}(x,t) \geq (\Delta_{g(t)} u)(x,t) + \langle X, \nabla u \rangle (x,t).$$

↳ (lower bounds are preserved for supersolutions).

<u>Theorem</u> M^n, $g(t)$, $X(t)$, $t \in [0,T)$

Let $\alpha \in \mathbb{R}$ s.t. $u(x,0) \geq \alpha$ $\forall x \in M^n$.

Also suppose that at any (x,t) when $u(x,t) < \alpha$, u is a supersolution to (1).

Then $u(x,t) \geq \alpha$ $\forall x \in M$, $t \in [0,T)$.

<u>Proof</u>: Suppose $H: M \times [0,T) \to \mathbb{R}$ and (x_0, t_0) is a point in $M \times [0,T)$ s.t.

H attains its minima along all points on M and earlier times

$$H(x_0,t_0) = \min_{M \times [0,t_0]} H(x,t).$$

$\Rightarrow$
$$\frac{\partial H}{\partial t}(x_0,t_0) \leq 0$$
$$\nabla H(x_0,t_0) = 0 \qquad \text{—— (2)}$$

$$\Delta H(x_0, t_0) \geq 0$$

define $H(x,t)$ by

$$H(x,t) = u(x,t) - \alpha + \epsilon t + \epsilon \quad , \epsilon > 0$$

$$H(x,0) = u(x,0) - \alpha + \epsilon > 0 \quad \text{at } t=0.$$

$$\frac{\partial H}{\partial t} = \frac{\partial u}{\partial t} + \epsilon$$

$$\nabla H = \nabla u, \qquad \Delta H = \Delta u$$

at any (x,t) where $u(x,t) < \alpha$

$$\frac{\partial H}{\partial t} \geq \Delta H + \langle x, \nabla H \rangle + \epsilon$$

To get, $u(x,t) \geq \alpha$ $\forall$ x, t, we'll prove $H > 0$. Suppose not.

Suppose $H(x_0, t_0) \leq 0$.

$\Rightarrow$ $\exists$ $x_1 \in M$ and $t_1 \in [0, t_0)$

s.t. $H(x_1, t_1) = 0$.

$$u(x_1, t_1) = \alpha - \epsilon t_1 - \epsilon < \alpha$$

$$0 \geq \frac{\partial H}{\partial t}(x_1, t_1) \geq \Delta H(x_1, t_1) + \langle X, \nabla H \rangle (x_1, t_1) + \epsilon \geq \epsilon > 0$$

contradiction.

$\therefore \quad H > 0 \implies u(x,t) \geq \alpha \quad \forall$
$x \in M, \; t \in [0, T]$.

$\square$

Heat-type eqn w/ linear reaction term

$$\frac{\partial u}{\partial t} = \Delta u + \underbrace{\langle X(t), \nabla u \rangle + F(u)}$$ reaction terms

F can be any non-linear function $F : \mathbb{R} \to \mathbb{R}$

$\beta : M \times [0, T) \longrightarrow \mathbb{R}$

$u(x,t)$ is a supersolution to

$$\frac{\partial v}{\partial t} = \Delta v + \langle X, \nabla v \rangle + \beta v$$

if

$$\frac{\partial u}{\partial t}(x,t) \geq (\Delta_{g(t)} u)(x,t) + \langle X, \nabla u \rangle (x,t) + (\beta u)(x,t).$$

<u>Theorem</u> :- Suppose that for each $\tau \in [0,T)$ $\exists$ a constant $C_\tau < \infty$ s.t. $\beta(x,t) \leq C_\tau$ $\forall$ $x \in M$ and $t \in [0,\tau]$.
If $u(x,0) \geq 0$ $\forall$ $x \in M$ then $u(x,t) \geq 0$ $\forall$ $x \in M$, $t \in [0,T)$.

<u>Proof</u> :- We want to define a new function which satisfies the heat eqn w/ a gradient term and thus reduces to the previous thm.

$\tau \in (0,T)$, define

$$J(x,t) = e^{-C_\tau t} u(x,t)$$

$$\frac{\partial J}{\partial t} = e^{-C_\tau t} \frac{\partial u}{\partial t} - C_\tau e^{-C_\tau t} u(x,t)$$

$$= e^{-C_\tau t} \left(\frac{\partial u}{\partial t} - C_\tau J \right)$$

$$\nabla J = e^{-C_\tau t} \nabla u$$

$$\Delta J = e^{-C_\tau t} \Delta u$$

$$\frac{\partial J}{\partial t} \geq \Delta J + \langle X, \nabla J \rangle + (\beta - C_\tau) J$$

$\because$ $\beta - C_\tau \leq 0$ $\forall$ $(x,t) \in M \times [0,\tau]$

$$\Rightarrow \frac{\partial J}{\partial t} \geq \Delta J + \langle X, \nabla J \rangle$$

$\forall$ $(x,t) \in M \times [0,\tau)$ s.t $J(x,t) \leq 0$.

$\therefore$ from the previous thm for $J(x,t)$

w/ $\alpha = 0$, $\left(\because J(x,0) = u(x,0) \geq \underset{\alpha}{\overset{\shortparallel}{0}}\right)$

we conclude that $J(x,t) \geq 0$ $\forall$ $x \in M$

and $t \in (0,\tau)$ $\Rightarrow$ $u \geq 0$ on $M \times [0,\tau)$.

$\because$ τ is arbitrary $\Rightarrow$ $u \geq 0$ on $M \times [0,T)$.

$\boxtimes$

<u>Application</u> Along the Ricci flow if $R(x,0) \geq 0$

then $R(x,t) \geq 0$ $\forall$ t s.t RF exists.

(see Pset 5).

(Heat-type eqn. non-linear reaction term).

(v^2)

$$\frac{\partial v}{\partial t} = \Delta v + \langle X, \nabla v \rangle + F(v)$$

$F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz.

$x \in \mathbb{R}$ then for every nbd $U \ni x$.

$$|F(y) - F(z)| \leq L_F |x-y|.$$

<u>Theorem</u> (ODE gives pointwise bounds for PDE).

Suppose $\exists$ $C_1 \in \mathbb{R}$ s.t. $u(x,0) \geq C_1$ $\forall$ $x \in M$. Let φ_1 be a solution to the associated ODE

$$\frac{d\varphi_1}{dt} = F(\varphi_1).$$

w/

$$\varphi_1(0) = C_1.$$

Then $u(x,t) \geq \varphi_1(t)$ $\forall$ $(x,t) \in M \times [0,T]$.

<u>Proof</u> <u>want</u> :- $u(x,t) \geq \varphi_1(t)$. $(u - \varphi_1 \geq 0)$

$$\frac{\partial}{\partial t}(u - \varphi_1) \geq \Delta(u-\varphi_1) + \langle X, \nabla(u-\varphi_1)\rangle + F(u) - F(\varphi_1)$$

notice that $u(x,0) \geq C_1 = \varphi_1(0)$

$\therefore \quad u(x,0) \geq \varphi_1$ at $t=0$.

Let $\tau \in (0,T)$. On $M \times [0,\tau]$ $\exists$

constants $C_\tau < \infty$ s.t. $|u(x,t)| \leq C_\tau$

$|\varphi_1(t)| \leq C_\tau$

$\because$ F is locally Lipschitz $\exists$ $L_\tau < \infty$

s.t.

$$|F(v)-F(w)| \leq L_\tau |v-w|$$

$$\forall \quad v, w \in [-C_\tau, C_\tau].$$

$\therefore$

$$\frac{\partial}{\partial t}(u-\varphi_1) \geq \Delta(u-\varphi_1) + \langle x, \nabla(u-\varphi_1)\rangle - L_\tau \underbrace{\text{signum}}_{\{-1,0,1\}}(u-\varphi_1)\cdot(u-\varphi_1)$$

$\because$ now we have a heat-type eqn w/ linear reaction term $\Rightarrow$ by the previous thm, $\Rightarrow$ $u(x,t) \geq \varphi_1(t)$ $\forall$ $(x,t) \in M \times [0,\tau]$

τ was arbitrary $\Rightarrow$ $u(x,t) \geq \varphi_1(t)$. $\boxtimes$