3 manifolds w/ positive Ricci-3

We have proved the pinching estimates and gradient estimates. Now we'll prove some other estimates:-

We have proved the following.

Let (M^3, g_0). Then RF exists for $t \in [0, T_{max})$

and $\limsup_{t \to T_{max}} |Rm| = \infty$.

In $n=3$ w/ $Ric(0) > 0 \Rightarrow Ric(t) > 0 \Rightarrow R(t) > 0$, we have $|Rm| \leq C|Rc|$ and $|Rc| \leq C' R$

$\therefore$ for $n=3$, $(M^n, g(0))$ w/ $R(0) > 0$, we get

$$\lim_{t \to T} R_{max}(t) = \infty.$$

<u>Lemma</u>:- Let $(M^3, g(t))$ be a RF w/ $Ric(0) > 0$. Then the solution becomes singular ie some finite time $T < \infty$. We also have:-

i) $\exists$ $C > 0$, $\gamma > 0$ depending only on g_0 s.t.

$$\frac{R_{min}}{R_{max}} \geq 1 - C R_{max}^{-\gamma} \quad \forall \; 0 \leq t < T.$$

In particular $R_{min}(x) = \infty$ as $t \to \Upsilon$.

ii) For $x \in M^3$ and $t \in [0, \Upsilon)$, let $\lambda(t) \geq \mu(t) \geq \nu(t)$ be the eigenvalues of the curvature operator. Then $\forall\ \epsilon \in (0,1)\ \exists\ \Upsilon_\epsilon \in [0, \Upsilon)$ s.t. $\forall\ \Upsilon_\epsilon \leq t < \Upsilon$, we have

$$\min_{x \in M} \nu(x,t) \geq (1-\epsilon) \max_{y \in M} \lambda(y,t) > 0.$$

$\therefore$ the sol^n eventually attains positive sectional curvature everywhere.

Proof:- By the gradient estimates, $\exists$ constant $A > 0$ and α s.t.

$$|\nabla R| \leq A\, R_{max}^{\frac{3}{2} - \alpha} \quad \forall\ t \in (\tau, T)$$

where $\tau \in [0, T)$.

Let $t \in (\tau, T)$. $\therefore$ M^3 is compact $\Rightarrow \exists\ \bar{x}(t) \in M^3$ s.t. $R_{max}(t) = R(\bar{x}, t)$. Given $\epsilon > 0$, consider the geodesic ball $B(\bar{x}, L)$ w/

$$L(t) = \frac{1}{\epsilon \sqrt{R_{max}(t)}}.$$

If γ is any minimizing geodesic from $\bar{x}$ to $x \in B(\bar{x}, L)$ then

$$R_{max} - R(x) \le \int_\gamma |\nabla R| \, ds \le A L R_{max}^{\frac{3}{2}-\alpha} \le \frac{A}{\epsilon} R_{max}^{1-\alpha}.$$

$\therefore$ on $B(\bar{x}, L)$ we have

$$R \ge R_{max}\left(1 - \frac{A}{\epsilon} R_{max}^{-\alpha}\right). \quad \text{—— } 1$$

$\therefore$ for some $\bar{t} \in (T_1 T)$ sufficiently close to T,

$$R \ge (1-\epsilon) R_{max} \quad \text{on } B(\bar{x}, L) \ \forall t \in [\bar{t}, T).$$

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Now, from the expression for $Rc = \frac{1}{2}\begin{pmatrix} \mu+\nu & & \\ & \lambda+\nu & \\ & & \lambda+\mu \end{pmatrix}$

in dim 3 and the pinching estimate
$\lambda \le C(\mu+\nu)$, we get that

$$Rc \ge \frac{\mu+\nu}{2} g \ge \frac{\lambda}{2C} g \ge \frac{\lambda+\mu+\nu}{6C} g = \frac{R}{6C} g$$

$$\ge \frac{R_{min}\, g}{6C}$$

$\therefore$ $Rc \ge 2\beta^2 R g$ for some $\beta > 0$. $\forall x \in M^3$.

Also, $\beta = \beta(g_0)$.

$\therefore$ By Myers's Thm (Suppose (M^n, g) complete, connected w/ $Rc \geq (n-1)Hg$ for some constant H. Then M^n is compact w/ finite fundamental group and diameter at most $\pi H^{-1/2}$.)

the minimizing geodesic γ from $\bar{x}$ must encounter a conjugate point within the distance $\dfrac{\pi}{\beta\sqrt{\inf_{B(\bar{x},L)} R}}$.

If $\epsilon > 0$ is sufficiently small then ② $\Rightarrow$

$$\frac{\pi}{\beta\sqrt{\inf_{B(\bar{x},L)} R}} \leq \frac{\pi}{\beta\sqrt{(1-\epsilon)R_{max}}} \leq \frac{1}{\epsilon\sqrt{R_{max}}} = L.$$

$\therefore$ $\operatorname{diam}(B(\bar{x},L)) \leq L$.

$\Rightarrow$ $B(\bar{x},L) = M$.

$\therefore$ $R \geq (1-\epsilon)R_{max}$ on $M \Rightarrow \dfrac{R_{min}}{R_{max}} \geq 1 - CR_{max}^{-a}$.

For 2), from the Ricci pinching improves result $\exists$

C and $\delta < 1$ s.t.

$$\lambda - \nu \leq C(\lambda+\mu+\nu)^{1-\delta} \Rightarrow \nu \geq \lambda - C(\lambda+\mu+\nu)^{1-\delta}$$

at all points on M.

$\therefore$ we have the pointwise inequality

$$\frac{\nu}{\lambda} \geq 1 - 3CR^{-\delta} \geq 1 - 3CR_{min}^{-\delta}.$$

Let $x, y \in M^3$ and $\eta > 0$ be given. Then $\exists\, T_\eta \in [0, T)$ s.t. $\forall\, T_\eta \leq t < T$ one has

$$\nu(x,t) \geq (1-\eta)\lambda(x,t)$$ (or can choose δ so that $3CR_{min}^{-\delta} < \eta$)

$$\geq \frac{1-\eta}{3} R(x,t)$$

$$\geq \frac{(1-\eta)^2}{3} R(y,t)$$ (from the estimate $R \geq (1-\epsilon)R_{max}$)

$$= \frac{(1-\eta)^2}{3} (\lambda+\mu+\nu(y,t))$$

$$\geq \frac{(1-\eta)^2}{3} (\lambda(y,t) + 2\nu(y,t))$$

$$\geq (1-\eta)^3 (\lambda(y,t))$$ by using the initial estimate $\nu \geq (1-\eta)\lambda$

Now take inf. over all $x \in M$ and sup. over all $y \in M$

to get $\min_{x \in M} \nu(x,t) \geq (1-\epsilon) \max \lambda(y,t) > 0$.

<u>Coroll</u> If $(M^3, g(t))$ is a sol^n on compact M^3 s.t. $Ric(g(0)) > 0$, then

$$\lim_{t \nearrow T} \sup \frac{|Rc - \frac{1}{3} R g|^2}{R^2} = 0.$$

$\square$

<u>Normalized Ricci flow</u>

$$\frac{\partial g}{\partial t} = -2 Ric + \frac{2}{n} \frac{\int R \, vol}{\int vol} g$$

$$\therefore \quad \frac{d}{dt}(vol) = -2R \, vol + 2 \frac{\int R \, vol}{\int vol} vol$$

and $\therefore$ vol(M) doesn't change along the flow.

<u>Lemma</u>:- RF and NRF are related to each other by a parametrization of space and time.

[illegible] Q 3 in PSET 2 whose solution has

Proof. This is a.s. ... been posted.

We'll prove the following lemma in the next lecture

Lemma:- $\tilde{R}_{max}$ is bounded for the normalized flow on M^3 w/ $Ric(g(0)) > 0$.

The proof of this lemma uses Bishop-Günther volume comparison theorem which we state below.

Bishop - Günther volume comparison Thm

Let $V_n^k(r)$ be the volume of the ball of radius r in the complete, simply-connected n-dimensional space of constant sectional curvature k. (so it is either S^n, $\mathbb{R}^n$ or $\mathbb{H}^n$). Let (M^n, g) be a Riemannian manifold, $p \in M$. Then

1) If $\exists$ $a > 0$ s.t. $Rc \geq (n-1)ag$ then

$$Vol(B(p,r)) \leq Vol_n^a(r).$$

2) If $\exists$ b s.t. all sectional curvatures of (M^n, g) ... by b and exp is injective on

are bounded above by

$B(p,r)$ then

$$\mathrm{Vol}(B(p,r)) \geq V_n^b(r).$$

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