Problem Set 7
Due date: 04.07.2022

Problems

(1) Prove that the set

$$K = \{M \mid \mu(M) + \nu(M) \geq 0\} \tag{0.1}$$

with M a curvature-like tensor, ν being the smallest eigenvalue and μ being the middle eignevalue, is preserved by the associated ODE. Conclude from this, using the maximum principle for systems, that non-negative Ricci curvature is preserved along the Ricci flow in dimension 3.

(2) Prove that the "Ricci pinching improves" estimate in dimension 3, i.e., existence of $C < \infty$ and δ such that $\lambda(M) - \nu(M) - C(\lambda + \mu + \nu)^{1-\delta} \leq 0$ implies that

$$\frac{\left|\mathrm{Ric} - \frac{1}{3}Rg\right|}{R} \leq CR^{-\delta} \tag{0.2}$$

and thus in regions where $R >> 0$, the manifold is almost Einstein.

(3) Prove the following evolution along the Ricci flow in dimension 3:

$$\frac{\partial}{\partial t}\left(|\mathrm{Ric}|^2 - \frac{1}{3}R^2\right) = \Delta\left(|\mathrm{Ric}|^2 - \frac{1}{3}R^2\right) - 2\left(|\nabla \mathrm{Ric}|^2 - \frac{1}{3}|\nabla R|^2\right) - 8\,\mathrm{tr}(\mathrm{Ric}^3) + \frac{26}{3}R|\mathrm{Ric}|^2 - 2R^3. \tag{0.3}$$

(4) First prove that in dimension 3, $\left|\nabla \mathrm{Ric} - \frac{1}{3}\nabla Rg\right|^2 \geq \frac{1}{3}\left|\mathrm{div}\left(\mathrm{Ric} - \frac{1}{3}Rg\right)\right|^2$. Then using this inequality and the contracted second Bianchi identity, prove that

$$|\nabla \mathrm{Ric}|^2 \geq \frac{37}{108}|\nabla R|^2. \tag{0.4}$$

Question to think: In the proof of the "Ricci pinching estimate" $\lambda(Rm) \leq C(\mu(Rm) + \nu(Rm))$ and the "Ricci pinching improves" $\lambda - \nu \leq C(\lambda + \mu + \nu)^{1-\delta}$, where did we use the fact that the initial Ricci curvature of M^3 is strictly positive (and not just non-negative)?