Problem Set 5
Due date: 27.06.2022

Problems

(1) Complete the proof of the following theorem from the lecture:

Let $g(t)$ be a solution of the Ricci flow and $i(t)$ be a solution of the system

$$\frac{\partial i(t)}{\partial t} = \mathrm{Rc} \circ i,$$
$$i(0) = i_0, \; i_0 : (V, h_0) \to (TM, g_0) \text{ is a bundle isometry}$$

then $i^*\mathrm{Rm}(X, Y, Z, W) = \mathrm{Rm}(i(X), i(Y), i(Z), i(W))$ evolves by

$$\frac{\partial R_{abcd}}{\partial t} = \Delta_D R_{abcd} + 2(B_{abcd} - B_{abdc} + B_{acbd} - B_{adbc}) \tag{0.1}$$

where Δ_D is the Laplacian as defined in the lecture and $B_{abcd} = -h^{eg}h^{fh}R_{aebf}R_{cgdh}$.

(2) Recall that for a Lie algebra $\mathfrak{g}$ with basis $\{\phi^\alpha\}$, a symmetric bilinear form $L \in \mathfrak{g} \otimes_S \mathfrak{g}$ can be described by its components $L_{\alpha\beta} = L(\phi^*_\alpha, \phi^*_\beta)$ and we get the operation $\#$ between two bilinear forms as

$$(L\#M)_{\alpha\beta} = C_\alpha^{\gamma\epsilon} C_\beta^{\delta\zeta} L_{\gamma\delta} M_{\epsilon\zeta}$$

where $C_\gamma^{\alpha\beta}$ are the structure constants defined by

$$\left[\phi^\alpha, \phi^\beta\right] = \sum_\gamma C_\gamma^{\alpha\beta} \phi^\gamma.$$

(a) Prove that $L\#M = M\#L$.
(b) Prove that if $L \geq 0$, then $L^\# = L\#L \geq 0$.

(3) Prove the following **doubling-time estimate**. For a solution $(M^n, g(t))_{t\in[0,T]}$ to the Ricci flow, there exists a constant $c = c(n) > 0$ such that

$$\sup_{x\in M} |\mathrm{Rm}(x,t)| \leq 2 \sup_{x\in M} |\mathrm{Rm}(x,0)| \qquad \text{for all times } t \in \Big[0, \min\Big\{T, \frac{c}{\sup_{x\in M} |\mathrm{Rm}(x,0)|}\Big\}\Big]. \tag{0.2}$$