# Pset 5 solutions

1.

(a) Recall that the evolution of the scalar curvature is

$$\partial_t R = \Delta R + 2|Ric|^2$$

Using the property that on an n-dim Riem. mfld, for a symmetric 2-tensor $A$, $|A|^2 \geq \frac{(trA)^2}{n}$,

we get that $|Ric|^2 \geq \frac{R^2}{n}$

$$\Rightarrow \partial_t R \geq \Delta R + \frac{2R^2}{n}$$

This eqn is of the form $\partial_t u \geq \Delta u + F(u)$ w/ $F(x) = \frac{2x^2}{n}$.

Thus, we can apply the max. principle for scalars w/ non-linear reaction term.
Clearly $R(0) \geq R_0 = \inf\{R(x,0)\}$.

So in order to get $R(t) \geq \phi(t)$, we need to solve the initial value problem

$$\frac{d\phi}{dt} = F(\phi)$$

$$\phi(0) = R_0$$

i.e.,

$$\frac{d\phi}{dt} = \frac{2}{n}\phi^2 \Rightarrow \phi^{-2} d\phi = \frac{2}{n} dt$$

$$\phi(0) = R_0.$$

Solving this ODE w/ the given initial value gives that $R(t) \geq \phi(t)$ w/

$$\phi(t) = \begin{cases} -\dfrac{n|R_0|}{n+2|R_0|t}, & R_0 < 0 \\ 0, & R_0 = 0 \\ \dfrac{nR_0}{n-2R_0 t}, & R_0 > 0. \end{cases}$$

(b) From the variation formulas for the volume form along the RF, we get

$$\frac{d}{dt}\mathrm{vol}_g = -R\,\mathrm{vol}_g$$

$$\Rightarrow \frac{d}{dt}\mathrm{Vol}(M, g(t)) = -\int R(x,t)\,\mathrm{vol}_{g(t)}.$$

$\because$ $R(x,0) \geq C \Rightarrow$ by the max. principle applied to $\partial_t R \geq \Delta R$ (which is still true)

we get $R(x,t) \geq C \implies -R(x,t) \leq -C$

$$\implies \frac{d}{dt} \mathrm{Vol}(M, g(t)) \leq -C \,\mathrm{Vol}(M, g(t))$$

Thus, solving this ODE gives

$$\mathrm{Vol}(M, g(t)) \leq e^{-Ct} \mathrm{Vol}(M, g(0))$$

Remark:- There was a typo in the problem; the RHS should be $e^{-Ct}$ and not $e^{-2Ct}$.

(2). Recall from Pset 4, Q.(2) that for $n=3$, we have

$$\partial_t R_{jk} = \Delta R_{jk} + 3RR_{jk} - 6R_{jp}R_{pk} + (2|Ric|^2 - R^2) g_{jk}$$

Comparing this w/ the statement for the max. principle for symmetric 2-tensors, we see that

$$\partial_t \alpha_{jk} = \Delta \alpha_{jk} + \beta_{jk}$$

w/ $\alpha_{jk} = R_{jk}$ and

$$\beta_{jk} = 3RR_{jk} - 6R_{jp}R_{pk} + (2|Ric|^2 - R^2) g_{jk}$$

To check the null eigenvector assumption, let $V$ be a null eigenvector of $Ric$, i.e.,

$R_{jk} V^k = 0$. Thus at the point in $M$ and at a time when $Ric$ has a null eigenvector, it can have at most two non-zero eigenvalue.

$\Rightarrow \quad |Ric|^2 \geq \frac{R^2}{2}$ at that point. (which is a better estimate than the Cauchy-Schwarz inequality $|Ric|^2 \geq \frac{R^2}{3}$).

So we get

$$\begin{aligned} B_{jk} V^j V^k &= 3R R_{jk} V^j V^k - 6 R_{jp} R_{pk} V^j V^k \\ &\quad + (2|Ric|^2 - R^2) g_{jk} V^j V^k \\ &= 0 - 0 + \underbrace{(2|Ric|^2 - R^2)}_{\geq 0 \text{ as } |Ric|^2 \geq \frac{R^2}{2}} |V|^2 \\ &\geq 0 \end{aligned}$$

$\therefore$ By the max. principle for symmetric 2-tensors,

$Ric(x,t) \geq 0 \quad \forall \; x \in M$ and $t$ if $Ric(x,0) \geq 0$.

$\square$

3. (a)

for $\epsilon = \frac{1}{3}$ ie $n = 3$, we actually get

$$Ric = \frac{1}{3} R g \Rightarrow g \text{ is Einstein} \Rightarrow$$

along RF, $g$ remains Einstein. $\Rightarrow R_{ij} \geq \frac{1}{3} R g_{ij}$ is preserved. So now let $\epsilon < \frac{1}{3}$.

We use the evolution of $R_{ij}$, $R$ and $g_{ij}$ to find the evolution of $R_{ij} - \epsilon R g_{ij}$. We get

$$\partial_t (R_{ij} - \epsilon R g_{ij}) = \Delta R_{ij} + 3 R R_{ij} - 6 R_{ip} R_{pj} + (2|Ric|^2 - R^2) g_{ij}$$
$$- \epsilon (\Delta R + 2|Ric|^2) g_{ij} + 2\epsilon R R_{ij}$$

now we manipulate the above expression to have $R_{ij} - \epsilon R g_{ij}$ in every term. We get

$$\partial_t (R_{ij} - \epsilon R g_{ij}) = \Delta (R_{ij} - \epsilon R g_{ij}) + 3R(R_{ij} - \epsilon R g_{ij})$$
$$- 6 (R_{ip} - \epsilon R g_{ip})(R_{pj} - \epsilon R g_{pj})$$
$$- 10 \epsilon R (R_{ij} - \epsilon R g_{ij})$$
$$+ \left(2(1-\epsilon)|Ric|^2 - (1 - 3\epsilon + 4\epsilon^2) R^2\right) g_{ij}$$

Also, note that if $R_{ij} \geq \epsilon R g_{ij}$ at $t=0$

$\Rightarrow$ $R_{ij} - \epsilon R g_{ij} \geq 0$ at $t=0$

$\Rightarrow$ $R - 3\epsilon R = (1-3\epsilon)R \geq 0$ at $t=0$

$\because$ $1-3\epsilon > 0$ $\Rightarrow$ $R \geq 0$ at $t=0$.

$\therefore$ By the max. principle, $R(t) \geq 0$ $\forall t$.

So comparing w/ the max. principle for symmetric 2-tensors, we have $\alpha_{ij} = R_{ij} - \epsilon R g_{ij}$ and

$$\beta_{ij} = 3R(R_{ij} - \epsilon R g_{ij}) - 6(R_{ip} - \epsilon R g_{ip})(R_{pj} - \epsilon R g_{pj})$$
$$- 10\epsilon R(R_{ij} - \epsilon R g_{ij}) + (2(1-\epsilon)|Rc|^2 - (1-3\epsilon+4\epsilon^2)R^2)g_{ij}$$

If at some point in $M$ and some time, $\alpha_{ij} V^j = 0$ for some $V$ and w/o loss of generality, let's assume that $|V|=1$, then we need to check that

$$\beta_{ij} V^i V^j \geq 0.$$

$\because$ $\alpha_{ij} V^j = 0$ $\Rightarrow$ at that point, the eigenvalues of $Ric$ are $\epsilon R, \lambda_1, \lambda_2$ w/ $\lambda_1 + \lambda_2 = (1-\epsilon)R$ as $\operatorname{tr} Ric = R$.

$$\therefore \beta_{ij} V^i V^j = 2(1-\epsilon)|Rc|^2 - (1-3\epsilon+4\epsilon^2)R^2$$
$$= 2(1-\epsilon)(\epsilon^2 R^2 + \lambda_1^2 + \lambda_2^2) - (1-3\epsilon+4\epsilon^2)R^2$$

$$= 2(1-\epsilon)(\lambda_1^2+\lambda_2^2) - (1-3\epsilon+2\epsilon^2+2\epsilon^3)R^2$$

$$\geq \left((1-\epsilon)^3 - (1-3\epsilon+2\epsilon^2+2\epsilon^3)\right)R^2$$

$$= (1-\epsilon^3-3\epsilon+3\epsilon^2-1+3\epsilon-2\epsilon^2-2\epsilon^3)R^2$$

$$= (\epsilon^2-3\epsilon^3)R^2 = \epsilon^2(1-3\epsilon)R^2 \geq 0$$

where for $\geq$ we used the fact that

$$\lambda_1+\lambda_2 = (1-\epsilon)R$$

$$\Rightarrow \quad (\lambda_1+\lambda_2)^2 = \lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2 = (1-\epsilon)^2R^2$$

and $\lambda_1^2+\lambda_2^2 \geq 2\lambda_1\lambda_2$

$$\Rightarrow \quad 2(\lambda_1^2+\lambda_2^2) \geq \lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2$$
$$= (\lambda_1+\lambda_2)^2 = (1-\epsilon)^2R^2.$$

Thus, applying the max. principle for symm. 2-tensors we get that $\alpha_{ij}(0)\geq 0 \Rightarrow \alpha_{ij}(t)\geq 0$

$$\Rightarrow \quad R_{ij} \geq \epsilon R g_{ij} \quad \forall \ \epsilon \leq \frac{1}{3} \quad \boxtimes.$$

(b) If you understood (a) then you should attempt this on your own w/ $\alpha_{ij} = \frac{1}{2}Rg_{ij} - R_{ij}$.