Problem Set 5
Due date: 30.05.2022

Problems

(1) (a) Using the maximum principle, prove that along the Ricci flow, if $R_0 = \inf\{R(x,0) \mid x \in M\}$ where R is scalar curvature then $R(x,t) \geq \phi(t)$ where

$$\phi(t) = \begin{cases} -\frac{-n|R_0|}{n+2|R_0|t} & \text{if } R_0 < 0, \\ 0 & \text{if } R_0 = 0, \\ \frac{nR_0}{n-2R_0 t} & \text{if } R_0 > 0. \end{cases}$$

Deduce that lower bound of the scalar curvature is preserved along the Ricci flow and in particular if at $t = 0$, $R \geq 0$ then it remains so for all time. This is an example of a **preserved curvature quantity along the Ricci flow.**

(b) Let $(M^n, g(t))_{t\in[0,T]}$ be a solution of the Ricci flow. For a constant C, suppose we have the pointwise bound for the scalar curvature $R(x,0) \geq C$ for all $x \in M$. Prove that

$$\mathrm{Vol}(M, g(t)) \leq e^{-2Ct}\mathrm{Vol}(M, g(0))$$

for all $t \in [0,T]$.

(2) Prove the following: Let $g(t)_{t\in[0,T)}$ be a solution to the Ricci flow in dimension 3. If $\mathrm{Ric}(0) > 0 (\geq 0)$ then $\mathrm{Ric}(t) > 0 (\geq 0)$ for all $t \in [0,T)$. This proves a theorem originally due to Hamilton in his "Three manifolds with positive Ricci curvature" paper and is another example of a preserved curvature quantity. **Remark:** The hypothesis $n = 3$ is crucial here. This result is false in higher dimensions.

(3) Prove the following **curvature pinching estimates** along the Ricci flow on a closed manifold in dimension 3.

(a) Prove that for any $\varepsilon \leq \frac{1}{3}$, the inequality $R_{ij} \geq \varepsilon R g_{ij}$ is preserved along the Ricci flow.

(b) Prove that the inequality $R_{ij} \leq \frac{1}{2} R g_{ij}$ is preserved along the Ricci flow.