Problem Session

① Given (M^n, g_0) $\exists$ sol^n to RF $(g(t))_{t\in[0,T)}$ on some maximal time interval.

(Later:- maximal time T is characterized by boundedness of $|Rm|$).

$$\mathcal{S} = \{ \text{RFs on } M^n \text{ w/ } g_0 \text{ as the initial condition} \}$$

$\neq \emptyset$

$$(g_t^1)_{t\in[0,T_1)} \leq (g_t^2)_{t\in[0,T_2)}$$

$$\iff T_1 \leq T_2 .$$

By Zorn's Lemma, $\exists$ a maximal element in $\mathcal{S}$, $\exists\, g(t)_{t\in[0,T_{max})}$.

T_{max} must be unique b/c maximality of the real number T_{max}. ☐

(3) X is conformal iff

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i = \frac{2}{n}(\mathrm{div} X) g_{ij} \quad \text{— (1)}$$

$Ric < 0$

Take div. of eq. (1)

$$\nabla_i (\nabla_i X_j + \nabla_j X_i) = \nabla_i \left(\frac{2}{n} \mathrm{div} X \, g_{ij} \right)$$

$$\Rightarrow \Delta X_j + \nabla_i \nabla_j X_i = \frac{2}{n} \nabla_j (\mathrm{div} X)$$

By Ricci identity

$$\Rightarrow \Delta X_j + \nabla_j \nabla_i X_i - R_{ijil} X_l = \frac{2}{n} \nabla_j (\mathrm{div} X)$$

$$\Rightarrow \Delta X_j + \nabla_j (\mathrm{div} X) + R_{jl} X_l = \frac{2}{n} \nabla_j (\mathrm{div} X)$$

$$\langle \Delta X, X \rangle = -Rc(X, X) + \left(\frac{2}{n} - 1 \right) \langle \nabla \mathrm{div} X, X \rangle$$

$$\frac{1}{2} \Delta |X|^2 = |\nabla X|^2 + \langle \Delta X, X \rangle$$

$$= |\nabla X|^2 - Rc(X,X) - \left(1-\frac{2}{n}\right)\langle \nabla(\operatorname{div} X), X\rangle$$

Integrate on M^n

$$g^{ij}\nabla_i(\operatorname{div} X) X_j$$

$$0 = \frac{1}{2}\int \Delta |X|^2 \, vol$$

$$= \int_M |\nabla X|^2 - Rc(X,X) - \left(1-\frac{2}{n}\right)\langle \overbrace{\nabla(\operatorname{div} X), X}\rangle \, vol$$

Integrate by parts to get

$$0 = \int_M \left(|\nabla X|^2 - Rc(X,X) + \left(1-\frac{2}{n}\right)|\operatorname{div} X|^2\right) vol$$

$$\int Rc(X,X)\, vol = \int |\nabla X|^2 + \left(1-\frac{2}{n}\right)|\operatorname{div} X|^2 \, vol$$

$$\Longrightarrow \quad Rc(X,X) = 0$$

$$\Longrightarrow \quad X = 0.$$

If X is a conformal v.f. on $Rc < 0$ manifold then $X = 0$. $\boxed{//}$

(b) $\int_M \langle \nabla R, X \rangle = \int_{M^n} R \operatorname{div} X = 0.$

X is conformal $\Rightarrow$

$$\nabla_i X_j + \nabla_j X_i = \frac{2}{n} (\operatorname{div} X) g_{ij}$$

Take inner product of this eqn with R_{ij}

$$\underbrace{R_{ij} (\nabla_i X_j + \nabla_j X_i)} = \frac{2}{n} (\operatorname{div} X) g_{ij} R_{ij}$$

$$= \cancel{2} R_{ij} \nabla_i X_j \qquad = \frac{\cancel{2}}{n} (\operatorname{div} X) R$$

$$\frac{1}{n} (\operatorname{div} X) R = R_{ij} \nabla_i X_j$$

Integrating on both sides

$$\boxed{\frac{1}{n} \int R (\operatorname{div} X) \, vol = \int_{M^n} R_{ij} \nabla_i X_j \, vol}$$

Integrate by parts and use twice contracted 2nd Bianchi

Exercise :-

$$\langle F, G \rangle = g^{ik} g^{jl} F_{ij} G_{kl}$$

$$g^{ik} g^{jl} R_{kl} \left(\nabla_i X_j + \nabla_j X_i\right) = \frac{2}{n}(\mathrm{div}\, X)\, g_{ij} R_{kl} g^{ik} g^{jl}$$

$$= \frac{2}{n}(\mathrm{div}\, X)\, R_{jl}\, g^{jl}$$

$$= \frac{2}{n}(\mathrm{div}\, X)\, R.$$

Ricci flow PSet 4 due 23.05.2022

(Thanks to Johannes)

② Decomposition

$$Rm = \frac{R}{2n(n-1)}(g \odot g) + \frac{1}{n-2}(\overset{\circ}{Ric} \odot g) + W.$$

For $n=3$:

$$R_{ijkl} = \frac{R}{12}(g \odot g)_{ijkl} + \left((Ric - \frac{R}{3}g) \odot g\right)_{ijkl}$$

Using the definition of $\odot$:

$$R_{pjkr} = \frac{R}{6}(g_{pr}g_{jk} - g_{pk}g_{jr}) + (R_{pr} - \frac{R}{3}g_{pr})g_{jk} + (R_{jk} - \frac{R}{3}g_{jk})g_{pr}$$
$$- (R_{pk} - \frac{R}{3}g_{pk})g_{jr} - (R_{jr} - \frac{R}{3}g_{jr})g_{pk}$$

$$(*) \quad \text{—} = -\frac{R}{2}g_{pr}g_{jk} + \frac{R}{2}g_{pk}g_{jr} + R_{pr}g_{jk} + R_{jk}g_{pr} - R_{pk}g_{jr} - R_{jr}g_{pk}$$

From the lectures we know that

$$\partial_t R_{jk} = \Delta R_{jk} + \underbrace{2g^{pq}g^{rs}R_{pjkr}R_{qs}} - 2g^{pq}R_{jp}R_{qk}.$$

Plugging $*$ in this yields

$$2g^{pq}g^{rs}R_{pjkr}R_{qs}$$
$$= 2g^{pq}g^{rs}\left(\frac{R}{2}(g_{pk}g_{jr} - g_{pr}g_{jk}) + R_{pr}g_{jk} + R_{jk}g_{pr} - R_{pk}g_{jr} - R_{jr}g_{pk}\right)R_{qs}$$
$$= 2g^{pq}g^{rs}R_{pr}R_{qs}g_{jk} - Rg^{pq}g^{rs}g_{pr}g_{jk}R_{qs} + Rg^{pq}g^{rs}g_{pk}g_{jr}R_{qs}$$
$$+ 2g^{pq}g^{rs}R_{jk}g_{pr}R_{qs} - 2g^{pq}g^{rs}R_{pk}g_{jr}R_{qs} - 2g^{pq}g^{rs}R_{jr}g_{pk}R_{qs}$$
$$= 2|Ric|^2 g_{jk} - Rg^q{}_r g^{rs}R_{qs}g_{jk} + Rg^q{}_k g_j{}^s R_{qs} + 2g^q{}_r g^{rs}R_{jk}R_{qs}$$
$$- 2g^{pq}R_{pk}R_{jk} - 2g^{pq}R_{pk}R_{jk}$$
$$= 2|Ric|^2 g_{jk} - Rg^{qs}R_{qs}g_{jk} + RR_{kj} + \underbrace{2g^{qs}R_{qs}R_{jk}}_{=2RR_{jk}}$$
$$- 4g^{pq}R_{qk}R_{jp}$$
$$= 2|Ric|^2 g_{jk} - R^2 g_{jk} + 3RR_{jk} - \underbrace{4g^{pq}R_{qk}R_{jp}}_{=4R_{pk}R_{pj}}$$

$$\longrightarrow \partial_t R_{jk} = \Delta R_{jk} - 6R_{pk}R_{pj} + 3RR_{jk} + (2|Ric|^2 - R^2)g_{jk}$$

$\boxtimes$