Ricci Flow – Problem Set 3

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Problem 1

a) First I show that g can be viewed as a steady Ricci soliton: This means I must find a vector field X such that $-2\operatorname{Ric}(g) = \mathscr{L}_X g$. As $\operatorname{Ric}(g) = 0$ one obvious solution is $X = 0$. However, other solutions exist as well: For every parallel vector field X one has

$$\begin{aligned}
\mathscr{L}_X g(v,w) &= X(g(v,w)) - g(\mathscr{L}_X v, w) - g(v, \mathscr{L}_X w) \\
&= \nabla_X g(v,w) + g(\nabla_X v, w) - g(v, \nabla_X w) - g(\mathscr{L}_X v, w) - g(v, \mathscr{L}_X w) \\
&= 0 + g(\nabla_v X, w) + g(v, \nabla_w X) \\
&= 0
\end{aligned}$$

where I used that ∇ is torsion-free.

Now I show that g can be regarded as an expanding gradient Ricci soliton with $\lambda = 1$ and $f(x) = \frac{1}{2}|x|^2$. That is, I must show that $-R_{ij} = \nabla_i \nabla_j f + \lambda g_{ij}$ where, due to the flatness of the Euclidean metric, $R_{ij} = 0$. This is a simple computation:

$$\begin{aligned}
\nabla_i \nabla_j f(x) &= \nabla_i (\partial_j \frac{1}{2}(x_1^2 + \cdots + x_n^2) \\
&= \nabla_i (x_j) \\
&= \delta_{ij} = g_{ij}
\end{aligned}$$

□

b) Suppose we have a self-similar solution

$$g(t) = \lambda(t) \phi_t^*(g_0).$$

Without loss of generality, assume that $\lambda(0) = 1$ and $\phi_0 = \mathrm{id}_M$. Since $g(t)$ is a solution to the Ricci flow, we get

$$-2\mathrm{Rc}(g_0) = \left.\frac{\partial g(t)}{\partial t}\right|_{t=0} = \lambda'(0) g_0 + \mathcal{L}_{Y(0)} g_0$$

and hence

$$-2\mathrm{Rc}(g_0) = 2\lambda g_0 + \mathcal{L}_X g_0$$

with $\lambda = \frac{1}{2}\lambda'(0)$ and $X = Y(0)$. Here $Y(t)$ is the vector field generating ϕ_t.

*Some parts are written by Shubham Dwivedi.

Conversely, suppose we have a Ricci soliton structure, i.e., $-2\mathrm{Rc}_0 = \mathcal{L}_X(g_0) + 2\lambda g_0$. Let

$$\lambda(t) = 1 + 2\lambda t, \quad Y_t(x) = \frac{X(x)}{\lambda(t)}$$

and consider $g(t) = \lambda(t)\phi_t^*(g_0)$ with ϕ_t the one-parameter family of diffeomorphisms generated by Y_t. We want to show that $g(t)$ is a solution ot the Ricci flow. We compute

$$\begin{aligned}
\partial_t g(t) &= \lambda'(t)\phi_t^*(g_0) + \lambda(t)\phi_t^*(\mathcal{L}_{Y_t} g_0) \\
&= 2\lambda\phi_t^*(g_0) + \phi_t^*(\mathcal{L}_X g_0) \\
&= \phi_t^*(2\lambda g_0 + \mathcal{L}_X g_0) \\
&= \phi_t^*(-2\mathrm{Rc}(g_0)) \\
&= -2\mathrm{Rc}(\lambda(t)\phi_t^*(g_0)) \\
&= -2\mathrm{Rc}(g(t)).
\end{aligned}$$

Problem 2

Suppose $g(t)$ is a gradient Ricci soliton, that is, $-R_{ij} = \nabla_i\nabla_j f + \lambda g_{ij}$ where $\lambda \in \mathbb{R}$ and $f \in C^\infty(M)$.

a) By tracing:

$$\begin{aligned}
& & -\lambda g_{ij} &= \nabla_i\nabla_j f + R_{ij} \\
&\iff & -\lambda g_{ii} &= R_{ii} + \nabla_i\nabla_i f \\
&\iff & -n\lambda &= R + \Delta f.
\end{aligned}$$

b) One has

$$0 = \nabla_j 0 = \nabla_j(R_{ij} + \nabla_i\nabla_j f + \lambda g_{ij}) = \nabla_j R_{ij} + \nabla_j\nabla_i\nabla_j f.$$

Apply the twice contracted second Bianchi identity (PSet 1):

$$\nabla_j R_{ij} = \frac{1}{2}\nabla_i R$$

And

$$\begin{aligned}
\nabla_j\nabla_i\nabla_j f &= \nabla_i\nabla_j\nabla_j f - R_{jijm}\nabla_m f \\
&= \nabla_i\Delta f + R_{im}\nabla_m f \\
&\overset{(a)}{=} \nabla_i(n\lambda - R) + R_{im}\nabla_m f \\
&= -\nabla_i R + R_{im}\nabla_m f.
\end{aligned}$$

Combining all results:

$$0 = \nabla_j R_{ij} + \nabla_j\nabla_i\nabla_j f = -\frac{1}{2}\nabla_i R + R_{im}\nabla_m f.$$

c) Using that $R_{ij} = -\nabla_i\nabla_j f - \lambda g_{ij}$, (b) becomes

$$\begin{aligned}
0 &= \nabla_i R - 2R_{im}\nabla_m f \\
&= \nabla_i R + (2\nabla_i\nabla_m f + 2\lambda g_{im})\nabla_m f \\
&= \nabla_i R + \nabla_i\nabla_m f\nabla_m f + \nabla_m f\nabla_i\nabla_m f + 2\lambda g_{im}\nabla_m f \\
&= \nabla_i R + \nabla_i|\nabla f|^2 + 2\lambda\nabla_i f \\
&= \nabla_i(R + |\nabla f|^2 + 2\lambda f).
\end{aligned}$$

Hence $R + |\nabla f|^2 - 2\lambda f$ is constant. □

Aside: I mentioned the following theorem of Perelman in the problem sessions.
Theorem(Perelman '03) Every compact Ricci soliton is a gradient Ricci soliton.

The proof of this theorem is beyond the scope of the lectures. If you are interested then you can have a look at Perelman's $\mathcal{W}-$functional and his proof of existence of minimization of the functional. The key ingredient is a logarithmic Sobolev estimate which we won't see in this course. However, there *do* exist noncompact Ricci solitons which are not gradient.

Also, by using the maximum principle (which we'll learn next week) and part (a) and (c) of Prob. 2, we can prove the following theorem. Note that for us, steady solitons means $\lambda = 0$ and expanding soliton means $\lambda < 0$ (due to the sign mistake in the assignment problem as we discussed)

Theorem. Every steady or expanding compact Ricci soliton has the scalar curvature R constant and in fact equal to $n\lambda$. Thus, from part (a), we get that $\Delta f = 0$ and hence f is a constant. Thus the Ricci soliton in this case is an Einstein metric.