Problem Set 2
Due date: 09.05.2022

Problems

(1) Let M^n be a closed manifold.

(a) Prove the **Bochner formula** for $|\nabla f|^2$, i.e., for $f \in C^\infty(M)$, prove that

$$\Delta|\nabla f|^2 = 2|\nabla\nabla f|^2 + 2R_{ij}\nabla_i f\nabla_j f + 2\nabla_i f\nabla_i(\Delta f). \tag{0.1}$$

Conclude from this that if $\mathrm{Ric} \geq 0$, $\Delta f = 0$ and $|\nabla f| =$ constant then ∇f is parallel.

(b) Prove the following integral equality:

$$\int_M |\nabla\nabla f|^2 \,\mathrm{vol} + \int_M \mathrm{Ric}(\nabla f, \nabla f)\,\mathrm{vol} = \int_M (\Delta f)^2\,\mathrm{vol} \tag{0.2}$$

and using the fact that[1], $|\nabla\nabla f|^2 \geq \frac{1}{n}(\Delta f)^2$, show that

$$\int_M \mathrm{Ric}(\nabla f, \nabla f)\,\mathrm{vol} \leq \frac{n-1}{n}\int_M (\Delta f)^2\,\mathrm{vol}\,. \tag{0.4}$$

(**Hint:** Integration by parts!)

(c) Use the above to prove the following theorem due to **Lichnerowicz.** Suppose f is an eigenfunction of Δ with eigenvalue $\lambda > 0$, i.e., $\Delta f + \lambda f = 0$. If $\mathrm{Ric} \geq (n-1)K$ for some constant $K > 0$ then $\lambda \geq nK$.

(2) (a) The purpose of this problem is to show that in dimension 3, the Ricci curvature determines the Riemann curvature tensor.
Let (M^3, g) be a 3-dimensional Riemannian manifold and let us diagonalize the curvature operator (as a self-adjoint operator on 2-forms) Rm with respect to a basis $\{e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$ of $\Lambda^2 T^*M$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of TM^3 (this is possible because Rm is self-adjoint). Suppose that, with respect to this basis, Rm is a diagonal matrix with entries $\lambda_1, \lambda_2, \lambda_3$ down the diagonal. Then with respect to the basis $\{e_1, e_2, e_3\}$, prove that the Ricci tensor takes the form

$$\mathrm{Ric} = \frac{1}{2}\begin{bmatrix} \lambda_2+\lambda_3 & 0 & 0 \\ 0 & \lambda_3+\lambda_1 & 0 \\ 0 & 0 & \lambda_1+\lambda_2 \end{bmatrix} \tag{0.5}$$

and the scalar curvature $R = \lambda_1 + \lambda_2 + \lambda_3$. (**Hint:** Use the geometric interpretation of the Ricci and scalar curvatures.)

(b) Prove that an Einstein metric on a manifold of dimension $n \geq 3$ has constant scalar curvature. If $n = 3$, the metric has constant sectional curvature.

(3) Instead of the Ricci flow, one can also look at the volume normalized version of the Ricci flow called the **normalized Ricci flow** which is the the following evolution equation for a family of metrics $g(t)$ on M^n:

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric} + \frac{2}{n}\frac{\left(\int_M R\,\mathrm{vol}\right)}{\left(\int_M \mathrm{vol}\right)}g(t) \tag{NRF}$$

where R is the scalar curvature. The advantage of (NRF) is that the volume of the evolving manifolds remains constant along (NRF). (You can check this one we derive the variational formula for the volume along a geometric flow of metrics). Prove that:

[1] This is the usual Cauchy-Schwarz inequality. More generally, if S is any $(2,0)-$tensor then

$$|S_{ij}|_g^2 \geq \frac{1}{n}(g^{ij}S_{ij})^2 \tag{0.3}$$

(a) A compact manifold (M^n, g) is a fixed point of (NRF) if and only if it is an Einstein manifold.

(b) Show that the unnormalized and normalized Ricci flows differ only by a rescaling of space and time. (You might've have to use the fact that if for a geometric flow $\partial_t g = h_{ij}$ for some symmetric 2-tensor h then $\partial_t \operatorname{vol}_{g(t)} = \frac{\operatorname{tr} h}{2} \operatorname{vol}$. We'll prove this in Monday's lecture but you can take it for granted for this problem.)