Problem Set 1
Due date: 02.05.2022

Problems

(1) Let M^n be a manifold. Recall the lemma about the induced connection on all tensor bundles from a connection ∇ on the tangent bundle (Lemma on pg. 15 and 16 of the lecture notes). Use the three properties mentioned in the lemma to prove the following:

(a) If α is a covector field and Y is a vector field then

$$\nabla_X \langle \alpha, Y \rangle = \langle \nabla_X \alpha, y \rangle + \langle \alpha, \nabla_X Y \rangle \tag{0.1}$$

where $\langle , \rangle$ is the natural pairing between a covector and a vector field.

(b) If ω is a 1-form then the coordinate expression of $\nabla_X \omega$ is given by

$$\nabla_X \omega = (X^i \partial_i \omega_k - X^i \omega_j \Gamma^j_{ik}) dx^k. \tag{0.2}$$

(2) Find the expression of the $(3,1)$ Riemann curvature tensor in coordinates in terms of the Christoffel symbols. We will use the expression when we will calculate the variation formulas under the Ricci flow. (**Hint:** Proceed in the same as we obtained a formula for the Christoffel symbols in terms of the derivatives of the components of the metric.)

(3) This question looks a bit longer because here we introduce the **Ricci curvature** and the **Scalar curvature** and prove some properties about them. Recall the following from linear algebra: If $(V, \langle , \rangle)$ is an inner product space with basis $\{e_1, \ldots, e_n\}$ and $A : V \to A$ is a linear map then $Ae_i = A^j_i e_j$ and $\mathrm{tr}(A) = A^i_i \in \mathbb{R}$. Note that

$$g^{ij} \langle Ae_i, e_j \rangle = g^{ij} \langle A^l_i e_l, e_j \rangle = g^{ij} A^l_i g_{lj} = A^i_i = \mathrm{tr}(A)$$

and thus $\mathrm{tr}(A) = g^{ij} \langle Ae_i, e_j \rangle$.
If B is a bilinear form then we define $\mathrm{tr}(B) = g^{ij} B_{ij}$.

Let (M, g) be a Riemannian manifold, $p \in M$ and fix $X_p, Y_p \in T_pM$. Define $A_p : T_pM \to T_pM$ by $A_p(Z_p) = R(Z_p, X_p)Y_p$, where R is the Riemann curvature tensor. Then from the above discussion we see that if $\{e_1, \ldots, e_n\}$ is any basis of T_pM then

$$\begin{aligned} \mathrm{tr}(A_p) &= g(A_p e_i, e_j) g^{ij}_p \\ &= g(R(e_i, X_p)Y_p, e_j) g^{ij} \\ &= R(e_i, X_p, Y_p, e_j) g^{ij}. \end{aligned}$$

We can now make the following definition

Definition The **Ricci tensor** of g is the $(2,0)$-tensor Ric defined by

$$\mathrm{Ric}(X, Y) = g^{ij} R(e_i, X, Y, e_j). \tag{0.3}$$

In local coordinates,

$$\mathrm{Ric} = R_{jk} dx^j \otimes dx^k \quad \textit{with} \quad R_{jk} = R_{ijkl} g^{il} \tag{0.4}$$

and thus Ric is obtained from the $(4,0)$-Riemann tensor by tracing on the first and the last index.
Definition The **scalar curvature** R of g is the trace of the Ricci tensor and is a function on M. That is, $\mathrm{R} = \mathrm{tr}_g(\mathrm{Ric}) = g^{ij} R_{ij}$.

Prove the following:

(a) Ric is a symmetric $(2,0)$-tensor.
(b) $\nabla_i R_{ijmk} = \nabla_k R_{jm} - \nabla_m R_{jk}$.
(c) $\nabla_i R_{ij} = \frac{1}{2} \nabla_j \mathrm{R}$ which in invariant form reads as $\mathrm{div}\,\mathrm{Ric} = \frac{1}{2} d\,\mathrm{R}$ where div is the divergence of a tensor.

(4) Let g be a Riemannian metric on a manifold M. Let $c > 0$ be a constant. Then cg is again a Riemannian metric on M as is said to be a rescaled metric. Show that as a $(3,1)$-tensor $R_{(3,1)}(cg) = R_{(3,1)}(g)$. Find the scalings of the $(4,0)$-Riemann curvature tensor, the Ricci curvature and the Scalar curvature. (**Hint:** What can you say about the Levi-Civita connection ∇^{cg} in terms of ∇^{g}?)