3 $\quad Ric = \dfrac{1}{n} \dfrac{\int R \, vol}{\int_M vol} g(t)$

$\Rightarrow \quad R = \dfrac{\int R \, vol}{\int vol}$ (by tracing)

$\Rightarrow dR = 0$ (by Stokes' theorem) $\Rightarrow R = \text{constant}$

$\therefore \quad Ric = \dfrac{\text{constant}}{n} g \Rightarrow$ Einstein manifold.

b

RF and NRF differ only by a scaling of space and time.

Let $g(t)$ be a RF. and suppose M has finite volume.

define dilations $\lambda(t) > 0$ s.t. $\tilde{g}(t) = \lambda(t) g(t)$

satisfy $\displaystyle\int_{M^n} \widetilde{vol}_{\tilde{g}(t)} = 1.$

Let $\displaystyle \tilde{t} = \int_0^t \lambda(\tau) \, d\tau.$

then $\dfrac{d\tilde{t}}{dt} = \lambda(t).$

Goal:- Show that

$$\frac{\partial \tilde{g}}{\partial \tilde{t}} = -2\widetilde{Ric}(\tilde{g}(t)) + \frac{2}{n} \frac{\int_M \tilde{R} \, \widetilde{vol}}{\int_M \widetilde{vol}} \tilde{g}(t)$$

we exactly know how geometric quantities scale as per the scaling of the metrics. (A1 last problem)

Then $\because$ $\frac{d}{dt}\int vol_{g(t)} = -\int R\, vol$

$\therefore$

$$\frac{\partial}{\partial \tilde{t}}\tilde{g}(t) = \frac{dt}{d\tilde{t}} \cdot \frac{\partial}{\partial t}\left(\lambda(t) g(t)\right)$$

$$= \frac{1}{\lambda(t)} \cdot \left(-2Rc(t)\cdot\lambda(t) + \partial_t \lambda(t)\, g(t)\right)$$

$$= -2\overline{Rc}(t) + \frac{\partial_t \lambda(t)}{\lambda^2(t)}\, \bar{g}(t) \quad \text{—— (1)}$$

Go back to the evolution eq^n of the vol. form and notice

$$\partial_t \log \det(\lambda(t) g(t)) = -2R + \frac{n}{\lambda(t)}\frac{d\lambda(t)}{dt}$$

and $\because$ $\int \widetilde{vol} = 1$ (by assumption)

$$\Rightarrow \quad 0 = \frac{d}{dt}\int \widetilde{vol} = \int \frac{\partial}{\partial t}\log\sqrt{\det(\lambda(t) g(t))}\,\widetilde{vol}$$

$$= \int\left(-\lambda\tilde{R} + \frac{n}{2\lambda}\frac{\partial\lambda}{\partial t}\right)\widetilde{vol}$$

$$= -\lambda\frac{\int \tilde{R}\,\widetilde{vol}}{\int\widetilde{vol}} + \frac{n}{2\lambda}\frac{\partial\lambda}{\partial t}\underbrace{\int \widetilde{vol}}_{=1}$$

$$\therefore \quad \frac{\lambda \int \tilde{R} \, \widetilde{vol}}{\int \widetilde{vol}} = \frac{n}{2\lambda} \frac{d\lambda}{dt}$$

$$\Rightarrow \quad \frac{2}{n} \frac{\int \tilde{R} \, \widetilde{vol} \, \tilde{g}(t)}{\int \widetilde{vol}} = \frac{1}{\lambda^2(t)} \frac{d\lambda}{dt} \tilde{g}(t) \quad \text{--- (2)}$$

$\therefore$ from (1) and (2) we get

$$\boxed{\frac{\partial \tilde{g}(t)}{\partial \tilde{t}} = -2\widetilde{Ric} + \frac{2}{n} \frac{\int \tilde{R} \, \widetilde{vol} \, \tilde{g}(t)}{\int \widetilde{vol}}}$$

→ so starting from a solⁿ $g(t)$ of the RF, by dilating space and time we got a solⁿ of the normalized Ricci flow.