Lecture 9

We'll start with the Logarithmic functions.

Logarithmic Functions

Logarithm is the inverse of exponential function.

Let $a > 0$, $a \neq 1$ as before and let $x > 0$, then

$$\boxed{y = \log_a x \iff a^y = x}$$

a is called the base of the logarithm.

Special Case :- If $a = e$ (the special constant we disc--ussed in Lecture 8) then we denote it by $\ln(x)$, i.e.,

$$\boxed{\ln x = \log_e x}$$ - call this "natural logarithm".

Thus we get a way to switch b/w exponential equa-tions and logarithm problems and to solve more exponential equations.

e.g. $3^2 = 9 \Rightarrow \log_3 9 = 2$

$e^x = 7 \Rightarrow \ln 7 = x$

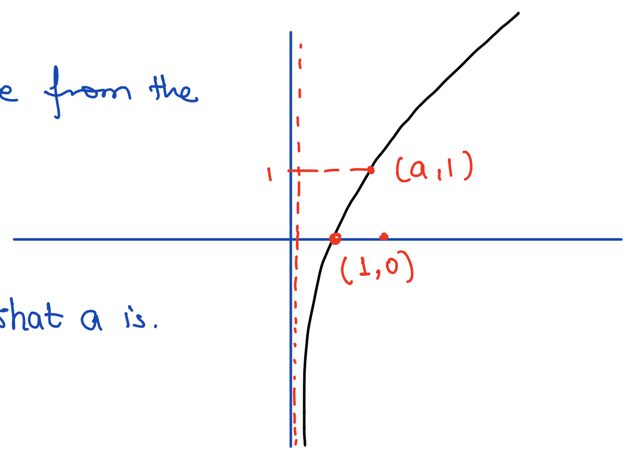
Domain :- $(0, \infty)$ [Recall the rules for finding the domain of a function]

Range :- $\mathbb{R}$

Vertical asymptote :- y-axis $(x = 0)$

The graph of $\log_a x$ looks like

As you will notice from the graph that $\log_a(1) = 0$, irrespective of what a is.



Properties of Logarithms

1) $\log_a(x \cdot y) = \log_a x + \log_a y$

2) $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$

3) $\log_a(x^r) = r \log_a x$

4) $\log_a 1 = 0$

5) $\log_a a = 1$ and hence $\ln e = 1$

6) $a^{\log_a x} = x$ and $\log_a a^x = x$

so, $e^{\ln x} = x$ and $\ln e^x = x$

Let's see how we can use properties of the logarithm to solve questions.

e.g. Simplify :- $\log_2 6 + \log_2 4 - \log_2 \left(\frac{3}{4}\right)$

Sol^n Use Prop. 1 and 2 to get

$$\log_2 6 + \log_2 4 - \log_2 (3/4) = \log_2 \left(\frac{6 \cdot 4}{3/4}\right)$$

$$= \log_2 \left(\frac{6 \cdot 4 \times 4}{3}\right) = \log_2 (32)$$

$$= \log_2 (2^5)$$

$$= 5 \log_2 2 \quad \text{(By Prop. 3)}$$

$$= 5 \quad (\text{as } \log_2 2 = 1 \text{ by Prop. 5}).$$

e.g. Simplify: $\ln(x^2) + \ln(x)$

$$= 2\ln(x) + \ln x = 3\ln x$$

Suppose we want to simplify $\log_2 6 - \log_4 9$.

Here the bases are different so we cannot apply any of the properties listed above.

Change of Base

Let $a, b, x > 0$, $a \neq 1$, $b \neq 1$. Then we have the change of base formula

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

We typically use ln's when working with this

$$\boxed{\log_a x = \frac{\ln x}{\ln a}}$$

Let's come back to the question above $\log_2 6 - \log_4 9$.

Change both the expressions in terms of ln by using the change of the base formula :-

$$\log_2 6 - \log_4 9 = \frac{\ln 6}{\ln 2} - \frac{\ln 9}{\ln 4} = \frac{\ln 6}{\ln 2} - \frac{\ln(3^2)}{\ln(2^2)}$$

$$= \frac{\ln 6}{\ln 2} - \frac{\cancel{2}\ln 3}{\cancel{2}\ln 2} = \frac{\ln 6 - \ln 3}{\ln 2}$$

$$= \frac{\ln(6/3)}{\ln 2} = \frac{\ln 2}{\ln 2} = 1$$

Solving Logarithmic Equations

We can use the properties of the logarithms to solve equations for x.

E.g. Solve for x.

1) $\log_x (8/27) = 3$

from the definition , $x^3 = \frac{8}{27} \Rightarrow x = \sqrt[3]{8/27}$

$$= \frac{2}{3}$$

2) $\log_4 (x) = 5/2 \Rightarrow 4^{\frac{5}{2}} = x$

$\Rightarrow (2^2)^{5/2} = x \Rightarrow 2^5 = x$

$\Rightarrow x = 32.$

3) $\log_2 8 = x \Rightarrow \log_2 2^3 = x \Rightarrow 3\log_2 2 = x$

$\Rightarrow 3 = x$ (as $\log_2 2 = 1$)

4) $\log_2(x) - \log_2(x-1) = 1$

By using prop. 2 we get

$$\log_2\left(\frac{x}{x-1}\right) = 1 \Rightarrow \frac{x}{x-1} = 2^1 \Rightarrow x = 2x-2$$

$$\Rightarrow \boxed{x=2}$$

Recall from Lecture 8 that we were unable to solve the equation $3^x = 5$ as they had different base. We can solve such equations now using logarithms. We need to remember the following rule

for $a > 0$, $a \neq 1$

$$\boxed{\log_a x = \log_a y \Leftrightarrow x = y}$$

↳ same base.

Let's look at $3^x = 5$. Take ln both sides to get

$$\ln(3^x) = \ln 5 \Rightarrow x\ln 3 = \ln 5 \Rightarrow \boxed{x = \frac{\ln 5}{\ln 3}}$$

e.g. Solve $3^{2x} = 4^{x+1}$

Take ln both sides

$$\ln(3^{2x}) = \ln(4^{x+1}) \Rightarrow 2x\ln 3 = (x+1)\ln 4$$

$$\Rightarrow x(2\ln 3 - \ln 4) = \ln 4$$

$$\Rightarrow \boxed{x = \frac{\ln 4}{2\ln 3 - \ln 4}}$$

Let's also note down an important formula which we'll use later.

$$\boxed{a^x = e^{x\ln a}}$$

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