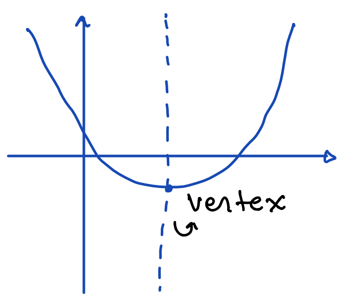
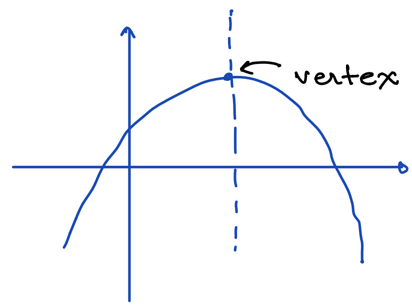
Lecture 6

Quadratic functions ; Translations & Reflections

A quadratic function has the form

$$y = ax^2 + bx + c \quad , \qquad a, b, c \in \mathbb{R}, \quad a \neq 0.$$



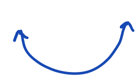
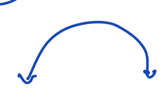


When working w/ quadratic functions, we should complete the square and write it as

$$\boxed{y = a(x-h)^2 + k} \qquad \text{———} \qquad (1)$$

which is more useful as it gives more information.
Graphs of quadratic functions are called parabolas.

- The point (h, k) is the vertex. This is the point where the graph has "peak / trough".

- If $a > 0$ then the graph opens upward. 
- If $a < 0$ then the graph opens downward. 

Then, finding the x & y intercepts and sketching will be easy!

The process of converting ax^2+bx+c in the form ① is called completing the square. Following are the steps to follow for this:-

Steps in completing the square	E.g. completing the square for $y = 2x^2 + 4x + 20$
(1) Factor the coefficient on x^2 from all x terms. We'll get something like $y = a(x^2+px)+q$.	$y = 2x^2+4x+20$ $\Rightarrow y = 2(x^2+2x)+20$ (i.e., $a=2$, $p=2$, $q=20$)
(2) Add and substract $\left(\frac{p}{2}\right)^2$ within the brackets. Remove the substracted term from the brackets.	Add and substract $\left(\frac{p}{2}\right)^2 = 1$. $y = 2(x^2+2x+1-1)+20$ $= 2(x^2+2x+1)-2+20$ $= 2(x^2+2x+1)+18$
(3) Factor the bracketed term as $\left(x+\frac{p}{2}\right)^2$. We're done!	$\frac{p}{2} = 1$ $\Rightarrow \boxed{y = 2(x+1)^2 + 18}$ Done!

Ques:- Complete the square for $y = -3x^2 + 5x + 6$.

Sol :- Let's follow the steps listed above.

1) $y = -3\left(x^2 - \frac{5}{3}x\right) + 6$.

$\Rightarrow \quad a = -3, \; p = -\frac{5}{3}, \; q = 6$.

2) $\frac{p}{2} = -\frac{\frac{5}{3}}{2} = -\frac{5}{6}$

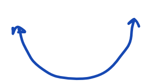
Thus, $y = -3\left(x^2 - \frac{5}{3}x + \frac{25}{36} - \frac{25}{36}\right) + 6$

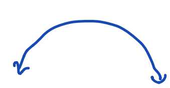
$\Rightarrow \quad y = -3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right) + \frac{25}{12} + 6$

3) $y = -3\left(x - \frac{5}{6}\right)^2 + \frac{97}{12}$

Graphing a Quadratic

To graph $y = a(x-h)^2 + k$

- plot vertex at (h, k).
- plot y-intercept which one gets by setting $x = 0$.
- plot x-intercept(s) which one gets by setting $y = 0$.
- Connect the points to make  If $a > 0$.

 If $a < 0$.

Remark :- The parabola may not have x-intercepts!
If $a \cdot k > 0$, then there are none.

E.g. Graph the quadratic $y = 2(x+1)^2 + 18$.

Solution The vertex is $(h,k) = (-1, 18)$ as the standard form is $y = a(x-h)^2 + k$.

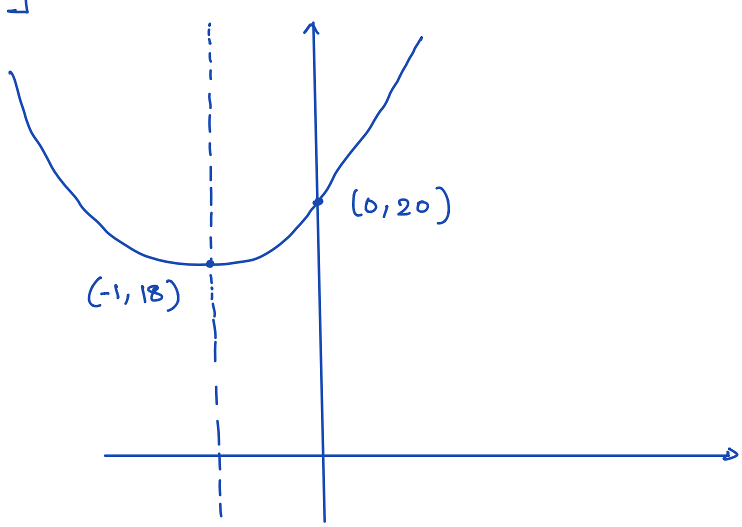
y-intercept. Set $x=0$ to get $y = 2\cdot(1)^2 + 18 = 20$.

So y-intercept is $(0, 20)$.

x-intercepts Note $a \cdot k = 2 \cdot 18 = 36 > 0 \Rightarrow$ there are no x-intercept.

[Note that when $y=0 \Rightarrow 2(x+1)^2 + 18 = 0 \Rightarrow (x+1)^2 = -9$ which can't happen as a square can't be negative.]

The graph is opening upward as $a = 2 > 0$.



E.g. Graph the quadratic $y = -(x-1)^2 + 4$.

solution. vertex $= (1,4)$

also $a = -1 \Rightarrow$ parabola opens down

y-intercept $x = 0 \Rightarrow y = -(-1)^2 + 4 = 3 \Rightarrow$

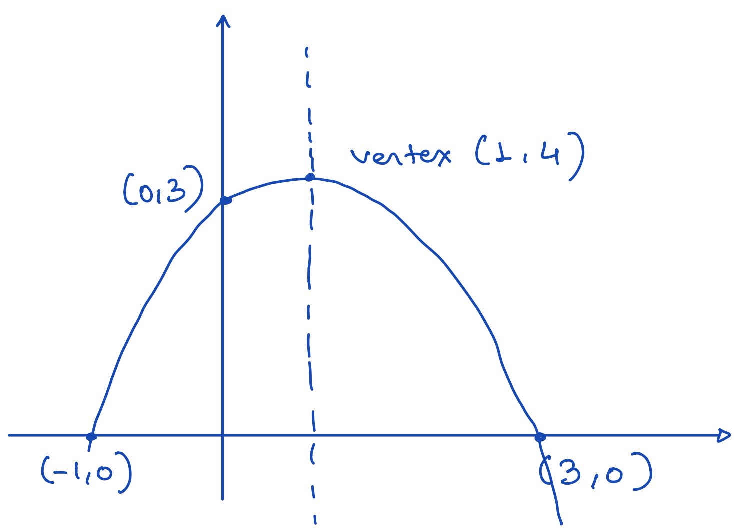
y-intercept is $(0,3)$

x-intercepts $\because$ Q.R $= -1 \cdot 4 = -4 \Rightarrow$ there are x-intercepts!

$y = 0 \Rightarrow -(x-1)^2 = -4 \Rightarrow (x-1)^2 = 4$

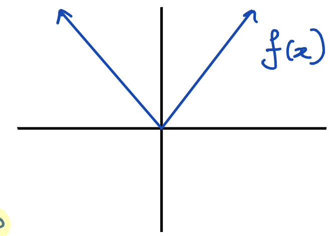
$\Rightarrow (x-1) = \pm\sqrt{4} = \pm 2 \Rightarrow x = 3, -1$

Thus x-intercepts are $(3,0)$ and $(-1,0)$.

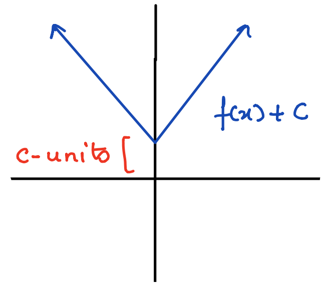


Translations & Reflections

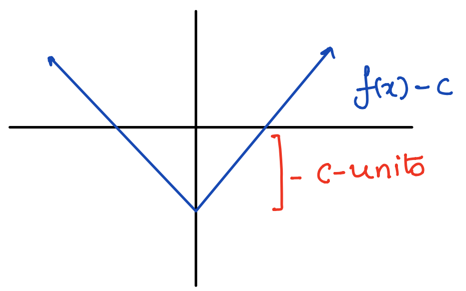
Suppose $f(x)$ is a function with graph and let $c > 0$ be a real numbers.



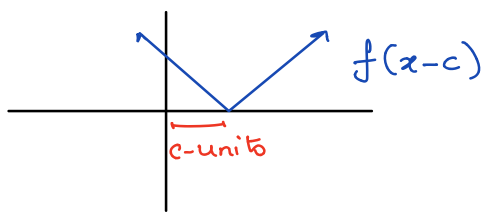
Then, $f(x)+c$ is the graph of $f(x)$ translated up by c-units. This can be seen from the fact that $f(x)+c$ just increases the value of $f(x)$ by c, for all x.



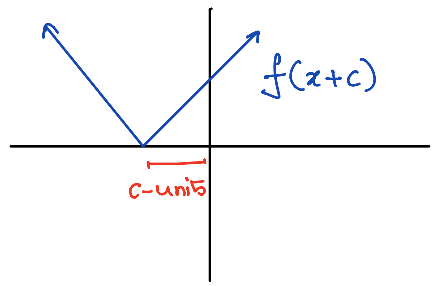
$f(x)-c$ is translated down by c-units.



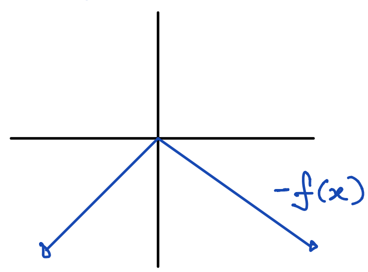
$f(x-c)$ is $f(x)$ translated right by c-units. This can be understood by observing that whatever value $f(x)$ was taking at x, $f(x-c)$ will take the same value at $x+c$. Thus, we must move the graph towards right.



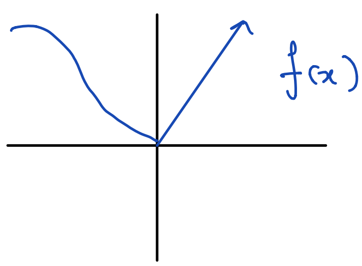
$f(x+c)$ is $f(x)$ translated left by c-units.

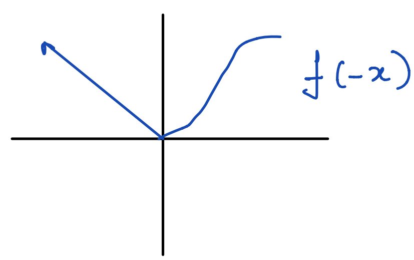


$-f(x)$ is $f(x)$ reflected over the x-axis, as $-f(x)$ will take the negative of the value which $f(x)$ took.



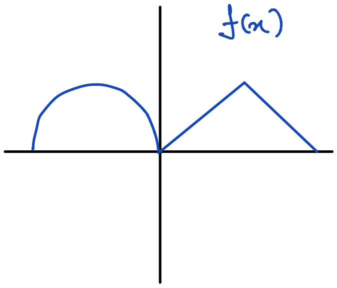
$f(-x)$ is $f(x)$ reflected over the y-axis.

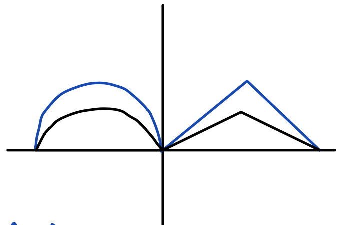




Multiplying $f(x)$ or x by a constant doesn't change the general shape of $f(x)$, but it compresses or stretches $f(x)$ horizontally or vertically.

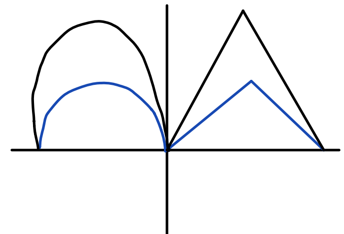
$c \cdot f(x)$ is a vertical stretch / compression.





blue = $f(x)$

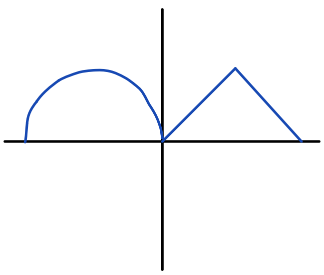
black - $\frac{1}{2} f(x)$ (i.e., $c = 1/2$)

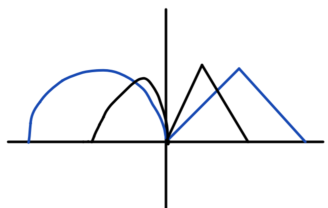


blue = $f(x)$

black - $2f(x)$ (i.e., $c=2$)

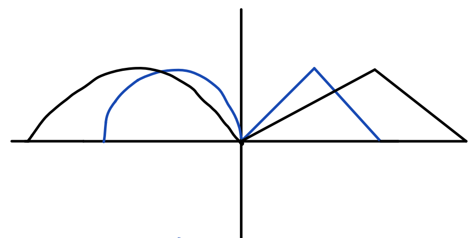
$f(c \cdot x)$ is a horizontal stretch / compression.





blue - $f(x)$

black - $f(c \cdot x)$, $c < 1$



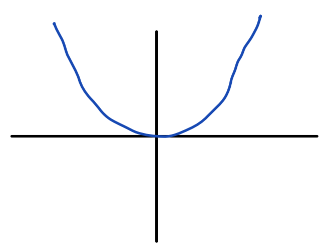
blue - $f(x)$

black - $f(c \cdot x)$, $c > 1$

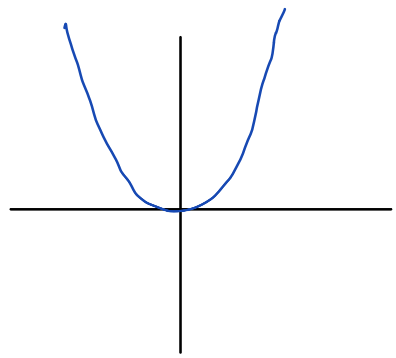
In practice, we will have to apply a combination of above transformations to sketch the graph of a function.

E.g. Sketch $-3x^2 + 2$.

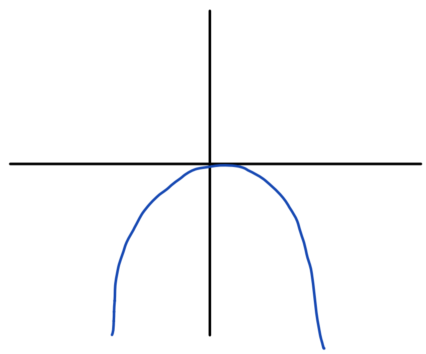
Sol. We start with $f(x) = x^2$



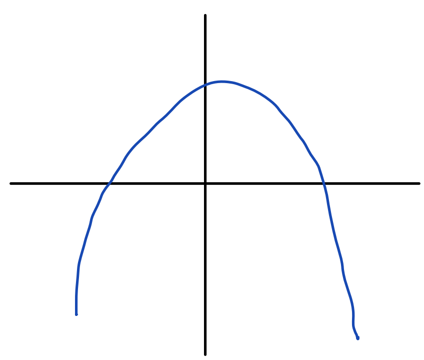
$3x^2$ stretches the graph in the vertical direction.



$-3x^2$ is reflection along the x-axis.



and finally $-3x^2+2$ is a translation upwards by 2 units.



$-3x^2+2$

which is the final answer.

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