Lecture 28

Higher Order derivatives ; concavity

Now that we know, how to find the derivative of a function, we can talk about higher order derivatives as well.

The 2^{nd} derivative of $f(x)$ denoted by

$f''(x)$ or $f^{(2)}(x)$ is

$$f''(x) = \left(f'(x)\right)'$$, i.e., first find $f'(x)$ and then differentiate that.

3^{rd} derivative of $f(x)$ is

$f'''(x) = f^{(3)}(x) = \left(f''(x)\right)'$, i.e., now differentiate the 2^{nd} derivative of $f(x)$.

$f''''(x) = f^{(4)}(x) = \left(f'''(x)\right)'$ is the 4^{th} derivative of $f(x)$ and so on.

e.g. If $f(x) = x^5 + 4x^4 + 2x^3 + 5x + 3$

Then $f'(x) = 5x^4 + 16x^3 + 6x^2 + 5$

$$f''(x) = (5x^4 + 16x^3 + 6x^2 + 5)'$$

$$= 20x^3 + 48x^2 + 12x$$

$$f'''(x) = (20x^3 + 48x^2 + 12x)'$$

$$= 60x^2 + 96x + 12$$

$$f''''(x) = (60x^2 + 96x + 12)'$$

$$= 120x + 96$$

...

② $f(x) = \cos x + \ln x$

$$f'(x) = -\sin x + \frac{1}{x}$$

$$f''(x) = \left(-\sin x + \frac{1}{x}\right)'$$

$$= -\cos x - \frac{1}{x^2}$$

$$f'''(x) = \left(-\cos x - \frac{1}{x^2}\right)'$$

$$= \sin x + \frac{2}{x^3} \quad \dots .$$

An application: If the distance is given by $f(t)$ then

$$\text{velocity} = f'(t)$$

$$\text{acceleration} = f''(t).$$

(Recall the last problem on the midterm).

Ques. If the distance covered by a runner is given by $f(t) = t^3 + 3t$, what is the acceleration at $t = 2s$.

Sol. $\text{acceleration} = f''(t)$

$$f'(t) = 3t^2 + 3$$

$$f''(t) = 6t$$

$\Rightarrow f''(2) = 12\, m/s^2 \Rightarrow$ the acceleration at 2 sec is $12\, m/s^2$.

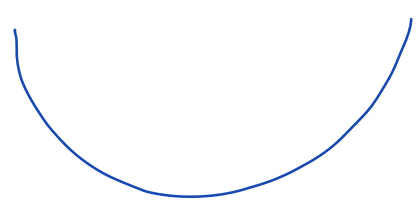
Concavity

Recall that $f'(x)$ is the rate of change of f.
Similarly, $f''(x)$ is the rate of change of f'.

Now when $f'(x) > 0 \Rightarrow f(x)$ is increasing
$f'(x) < 0 \Rightarrow f(x)$ is decreasing

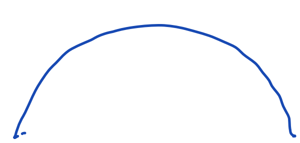
So $f''(x) > 0 \Rightarrow f'(x)$ is increasing
$f''(x) < 0 \Rightarrow f'(x)$ is decreasing.

Thus when $f''(x) > 0$ then $f'(x)$ is increasing
$\Rightarrow$ the slope of the tangent line to $f(x)$ is increasing $\Rightarrow f(x)$ looks like



— concave up

- slope increasing

Similarly when $f''(x) < 0 \Rightarrow f'(x)$ is decreasing
$\Rightarrow f(x)$ looks like  concave down.

Thus

$f(x)$ is concave up on an interval I if $f''(x) > 0$ on I.

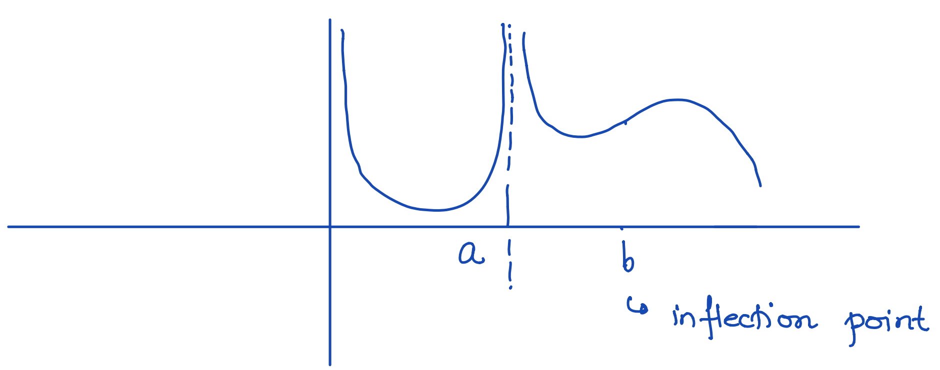
$f(x)$ is concave down on I if $f''(x) < 0$ on I.

Defn A point c in the domain of $f(x)$ is called an inflection point if the concavity changes at c, i.e, $f(x)$ becomes concave down from concave up or vice-versa.

If c is an inflection point $\Rightarrow f''(c) = 0$ or $f''(c)$ DNE.

(compare this with critical point and $f'(x)$.).

e.g.



Ques. find the intervals of concavity for

$f(x) = 3x^5 + 5x^4 - 20x^3 + 4$

Sol^n:- $f'(x) = 15x^4 + 20x^3 - 60x^2$

$$f''(x) = (f'(x))'$$
$$= 60x^3 + 60x^2 - 120x$$
$$= 60x(x^2 + x - 2)$$
$$= 60x(x+2)(x-1)$$

$f''(x)$ exists everywhere and $f''(x) = 0$

at $x = 0$, $x = -2$ and $x = 1$

$\therefore$ These are the inflection points.

If $x \in (-\infty, -2) \Rightarrow f''(x) < 0 \Rightarrow$ concave down

If $x \in (-2, 0) \Rightarrow f''(x) > 0 \Rightarrow$ concave up

If $x \in (0, 1) \Rightarrow f''(x) < 0 \Rightarrow$ concave down

If $x \in (1, \infty) \Rightarrow f''(x) > 0 \Rightarrow$ concave up

Ques find the intervals of concavity of $f(x) = \frac{1}{x}$

Soln $f'(x) = -\frac{1}{x^2}$

and $f''(x) = \frac{2}{x^3}$

$\therefore$ $f''(x)$ is never zero and $f''(x)$ DNE at $x=0$.

However $x=0$ is NOT in the domain $\Rightarrow$ $x=0$ is not an inflection point.

If $x \in (-\infty, 0) \Rightarrow f''(x) < 0 \Rightarrow$ concave down

$x \in (0, \infty) \Rightarrow f''(x) > 0 \Rightarrow$ concave up.

We'll use the concavity for curve sketching in the next lecture.

Before that, let's see how to use $f''(x)$ for finding local max/min.

Second Derivative Test for local Max/min

Let c be a critical point of $f(x)$ ($f'(c)=0$ or $f'(x)$ DNE at c)

Then if $f''(c) > 0 \Rightarrow c$ is a local min

$f''(c) < 0 \Rightarrow c$ is a local max.

Ques find the critical points and local max/min of $f(x) = x^3 - 3x^2 - 9x + 1$.

Sol^n For critical points

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 0$$

$$\Rightarrow 3(x-3)(x+1) = 0$$

$\Rightarrow x = 3$ and $x = -1$ are the critical points as $f'(x)$ exists everywhere.

To check local max/min, we use the 2nd derivative test.

$$f''(x) = 6x - 6 = 6(x-1)$$

$f''(3) = 6(3-1) = 12 > 0 \Rightarrow x = 3$ is a local min.

$f''(-1) = 6(-1-1) = -12 < 0 \Rightarrow x = -1$ is a local max.

Remark :

1. You can also use the first derivative test for local max/min. You will get the same answer.

2. Be careful with the 2^{nd} Derivative Test

$f''(c) > 0 \implies$ local min. (and NOT local max.)

$f''(c) < 0 \implies$ local max.