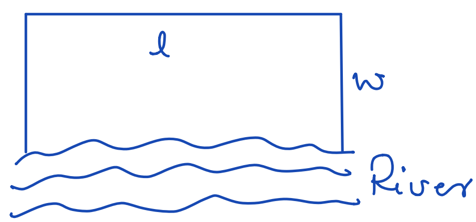
Lecture 26

Optimization Problems (Not in Textbook)

Now that we know the method to find the absolute max/min of a function, we can use this to solve real world problems.

E.g. A farmer has 800 m of fencing and wants to fence a field. Assuming one side of the field is a river, find the dimensions of the rectangular field that gives the largest area.

Sol.



The situation of the problem is as shown in the figure.

The Area $= l \cdot w$

$\because$ The farmer has 800 m of fencing and one side is river

$$\Rightarrow \quad l + 2w = 800$$

$$\Rightarrow \quad l = 800 - 2w$$

$\therefore$ Area $= (800-2w)\,w = 800w - 2w^2$

$\because$ we want to maximize the area $\Rightarrow$ we want to find w s.t. Area is maximum.

Clearly $w \geq 0$. Now $\because$ $l + 2w = 800$ $\Rightarrow$ the maximum amount w could be is 400 $\Rightarrow$ $w \in [0, 400]$.

We have $(\text{Area})' = 800 - 4w = 0 \Rightarrow w = 200$

$\therefore$ $w = 200$ is the critical point.

We now compute

$$A(0) = 0$$

$$A(200) = 800 \times 200 - 2 \cdot (200)^2$$

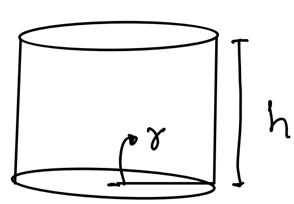
$$= 160000 - 2 \cdot 40000 = 80000$$

$$A(400) = 800 \times 400 - 2 \cdot (400)^2 = 0$$

$\therefore$ the maximum Area is $80{,}000 \, m^2$.

<u>Ques</u> Suppose we have 300 cm^2 of tin to work with and we want to make the biggest, most awesome soup can ever. How much soup could our can hold?

<u>Sol^n</u>



The situation is shown in the diagram.

We want to maximize the volume

$$v = \pi r^2 h$$

The amount of tin available $= 300\ cm^2$

$$\Rightarrow \underbrace{\pi r^2}_{\text{top}} + \underbrace{\pi r^2}_{\text{bottom}} + \underbrace{2\pi r h}_{\text{side}} = 300$$

$$\Rightarrow 2\pi r^2 + 2\pi r h = 300 \Rightarrow h = \frac{300 - 2\pi r^2}{2\pi r}$$

$$= \frac{150}{\pi r} - r$$

$$\Rightarrow \text{vol} = \pi r^2 \left(\frac{150}{\pi r} - r \right) = 150 r - \pi r^3$$

We want to maximize vol. So we need bounds on r.

Clearly $r \geq 0$.

Also $\because$ we have $300\ cm^2$ of tin, we must have

$$2\pi r^2 \leq 300 \Rightarrow r^2 \leq \frac{150}{\pi}$$

$$\Rightarrow r \leq \sqrt{\frac{150}{\pi}}$$

$$\therefore \quad 0 \leq r \leq \sqrt{\frac{150}{\pi}}$$

$\therefore \quad (\mathrm{vol})' = 150 - 3\pi r^2 = 0$

$\Rightarrow \quad \pi r^2 = 50 \quad \Rightarrow \quad r = \sqrt{\frac{50}{\pi}}$

So we compute $\quad \mathrm{Vol}(0) = 0$

$$\mathrm{Vol}\left(\sqrt{\frac{150}{\pi}}\right) = 0$$

$$\mathrm{Vol}\left(\sqrt{\frac{50}{\pi}}\right) = \pi\left(\sqrt{\frac{50}{\pi}}\right)^2\left(\frac{150}{\pi \cdot \sqrt{\frac{50}{\pi}}} - \sqrt{\frac{50}{\pi}}\right)$$

$$= \pi \cdot 500\left(\frac{150}{\sqrt{50\pi}} - \sqrt{\frac{50}{\pi}}\right)$$

$$\simeq 399 \text{ cm}^3$$

$\therefore$ the maximum amount of soup which the can can hold is 399 cm^3.

Strategy to solve optimization problems

① Understand the situation of the problem. Drawing a picture is often helpful.

② Identify the quantity to be maximized or minimized. Find an expression for that quantity.

(3) Use other info in the problem to get the formula from (2) into a function with one variable.

(4) Find the domain or bounds of the variable.

(5) Find the global max/min of the function on the domain.