Lecture 22

Implicit Differentiation (Not in textbook)

Up till now, we were given an expression for a function $y = f(x)$ explicitly, and then we differentia-ted it. But what if the expression of y is given implicitly?

e.g. (1) $x^2 + y^2 = 1$

(2) $2xy + 3x^2y^2 = e^x$

(3) $\ln(xy) + 2y^2 = \sin(y)$

In all the above examples, we do not have an explicit expression as $y =$ "something". Rather, we have an expression which involves both x (independent variable) and y (dependent variable) and we want to find $\frac{dy}{dx}$.

To find $\frac{dy}{dx}$, we follow what is known as <u>implicit differentiation</u>.

We follow the chain rule and remember that

y is a function of x

Thus, wherever we see an expression involving y, we differentiate it using the same rules as before, but we also write y' or $\frac{dy}{dx}$ with it as we are just following the chain rule.

Find y' or $\frac{dy}{dx}$ in the examples above.

1. $$x^2 + y^2 = 1$$

Differentiating both sides give

$$2x + \underbrace{2y \cdot \frac{dy}{dx}}_{} = 0$$

chain rule for y^2. ($2y$ from the power rule and $\frac{dy}{dx}$ because of the chain rule.)

$$\Rightarrow \quad 2\left(x + y\frac{dy}{dx}\right) = 0 \quad \Rightarrow \quad y\frac{dy}{dx} = -x$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

(just solve for $\frac{dy}{dx}$)

2. $2xy + 3x^2y^2 = e^x$

We differentiate everything and use the product rule for both the terms on the LHS.

$$\underbrace{2y + 2x\frac{dy}{dx}}_{\frac{d}{dx}(2xy)} + \underbrace{6xy^2 + 6x^2y\frac{dy}{dx}}_{\frac{d}{dx}(3x^2y^2)} = \underbrace{e^x}_{\frac{d}{dx}(e^x)}$$

now we solve for $\frac{dy}{dx}$ to get

$$\frac{dy}{dx}\left(2x + 6x^2y\right) = e^x - 2y - 6xy^2$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{e^x - 2y - 6xy^2}{2x + 6x^2y}}$$

3. $\ln(xy) + 2y^2 = \sin(y)$

We differentiate both sides and use chain rule to get

$$\frac{1}{xy}\cdot\frac{d}{dx}(xy) + 4yy' = \cos y \cdot y'$$

$$\Rightarrow \frac{1}{xy}\left(y + xy'\right) + 4yy' = \cos y \cdot y'$$

$$\Rightarrow \frac{1}{x} + \frac{y'}{y} + 4yy' = \cos y \, y'$$

$$\Rightarrow \frac{1}{x} = y'\left(\cos y - \frac{1}{y} - 4y\right)$$

$$\Rightarrow \boxed{y' = \frac{1}{x\left(\cos y - \frac{1}{y} - 4y\right)}}$$

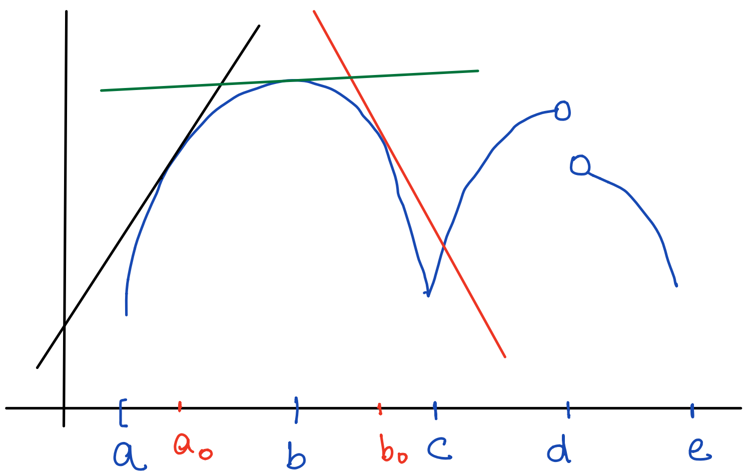
Applications of Derivatives

We'll now see the applications of derivatives. The topics in this section will include

- increasing / decreasing functions
- maxima / minima
- Optimization problem / Real-world problems
- related rates
- curve sketching

Increasing / Decreasing functions

We start by observing the following graph of a function.



Observations :→

① In the interval $[a,b]$ the value of the function is increasing as the value of x is increasing.

② The slope of the tangent line at any point in $[a,b]$ is positive. e.g. at a_0, the slope of the tangent line is positive.

③ In $[b,c]$ the value of the function is decreasing as x is increasing and the slope of the tangent line at any point (e.g. at b_0) is negative.

④ The values of the function starts decreasing after increasing at the point "b" and the tangent line is horizontal at b, i.e., the slope = 0.

⑤ The function again starts increasing from point "c" onwards and the derivative at "c" DNE.

⑥ The function is undefined at "d" and the behaviour of the function again changes from increasing to decreasing at the point "d".

Again, recall that slope of the tangent line at x being positive (respectively negative) means that
$$f'(x) > 0 \text{ (respectively } f'(x) < 0\text{)}.$$

Def^n:- A function $f(x)$ on an interval I is said to be

- increasing if $\forall\, x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
- decreasing if $\forall\, x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Thus, the function whose graph is given above is

- increasing ie $[a, b]$ and $[c, d]$
- decreasing ie $[b, c]$ and $[d, e]$

In the next lecture, we will learn the test to find the intervals where the function is increasing/decreasing.