Lecture 20

We now want to find the derivative of $f(x) = \log x$.

We observe that $e^{\ln x} = x$

so differentiating both sides give

$$\frac{d}{dx}\left(e^{\ln x}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \quad e^{\ln x} \cdot \frac{d}{dx}(\ln x) = 1 \qquad \text{(Chain rule)}$$

$$\Rightarrow \quad x \cdot \frac{d}{dx}(\ln x) = 1$$

$$\Rightarrow \quad \boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \quad , x > 0.$$

<u>Ques</u> find the derivative of the following functions.

① $f(x) = \ln(3x^2+2)$.

<u>Sol</u> We use chain rule as $\ln(3x^2+2) = g(h(x))$ with

$$g(x) = \ln(x) \quad \text{and} \quad h(x) = 3x^2+2$$

and $\because$ $g'(x) = \frac{1}{x}$ and $h'(x) = 6x$

$\therefore \quad \frac{d}{dx}\left(\ln(3x^2+2)\right) = g'(h(x)) \cdot h'(x)$

$$= \frac{1}{3x^2+2} \cdot (6x) = \frac{6x}{3x^2+2}$$

(2) $f(x) = \dfrac{x^2+2x}{\ln x}$

Sol We use the quotient rule to get

$$f'(x) = \frac{\frac{d}{dx}(x^2+2x) \cdot \ln x - \frac{d}{dx}(\ln x) \cdot x^2+2x}{(\ln x)^2}$$

$$= \frac{(2x+2)\ln x - \frac{1}{x} \cdot (x^2+2x)}{(\ln x)^2}$$

$$= \frac{(2x+2)\ln x - (x+2)}{(\ln(x))^2} \quad \text{Ans}$$

What about logs with different base?

Recall from the change of base formula, we get

$$\log_a x = \frac{\ln x}{\ln a}$$ and since $\ln a$ is a constant

$$\Rightarrow \frac{d}{dx}\left(\log_a x\right) = \frac{1}{\ln a} \cdot \frac{d}{dx}\left(\ln x\right)$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \cdot \ln a}}$$

<u>Ques</u> find the derivative of $f(x) = 2^x \cdot \log_2 (x^2-2)$.

<u>Sol</u> We first use the product rule to get

$$\frac{d}{dx}\left(2^x \cdot \log_2 (x^2-2)\right) = \frac{d}{dx}(2^x) \cdot \log_2 (x^2-2) + 2^x \cdot \frac{d}{dx}\left(\log_2 (x^2-2)\right)$$

$$= 2^x \cdot \ln 2 \cdot \log_2 (x^2-2) + \underbrace{2^x \cdot \frac{1}{(x^2-2) \cdot \ln 2} \cdot \frac{d}{dx}(x^2-2)}_{\text{Chain rule here}}$$

$$= 2^x \cdot \ln 2 \cdot \log_2 (x^2-2) + \frac{2^x \cdot (2x)}{(x^2-2) \ln 2}$$ Ans

Logarithmic Differentiation

We know how to differentiate $f(x)^n$ (use chain rule and power rule) and $a^{f(x)}$ (use chain rule and derivative of exponential function).

In the first case, the base is a function and the exponent is a constant and in the second case, the exponent is a function and the base constant.

But what about functions of the form $f(x)^{g(x)}$?

e.g. $f(x) = 3x^{2x^2+1}$ or $f(x) = \ln x^{\ln x}$

We use what is called logarithmic differentiation.

Here is how to do this with an illustration.

Steps for logarithmic Differentiation

Steps.	e.g. say $f(x) = x^x$
① Take ln of both sides	$f(x) = x^x$ $\Rightarrow \ln f(x) = \ln(x^x)$ $= x \ln x$
② Differentiate both sides.	$\frac{d}{dx}(\ln f(x)) = \frac{d}{dx}(x \ln x)$ $\Rightarrow \frac{f'(x)}{f(x)} = \ln x + 1$
③ Solve for $f'(x)$.	$f'(x) = f(x)(\ln x + 1)$ $= x^x(\ln x + 1)$ (Answer)

e.g. Suppose $f(x) = (x^2+1)^{e^x}$. Find $f'(x)$.

① $\ln f(x) = \ln\left((x^2+1)^{e^x}\right) = e^x \ln(x^2+1)$

$$\Rightarrow \frac{f'(x)}{f(x)} = \frac{d}{dx}(e^x) \cdot \ln(x^2+1) + e^x \cdot \frac{d}{dx}\left(\ln(x^2+1)\right)$$

$$= e^x \cdot \ln(x^2+1) + e^x \cdot \frac{1}{x^2+1} \cdot (2x)$$

$$= e^x\left(\ln(x^2+1) + \frac{2x}{x^2+1}\right)$$

$$\Rightarrow \quad f'(x) = f(x) \cdot \left(e^x \left(\ln(x^2+1) + \frac{2x}{x^2+1} \right) \right)$$

$$= (x^2+1)^{e^x} \cdot \left(e^x \left(\ln(x^2+1) + \frac{2x}{x^2+1} \right) \right)$$

<u>Ans</u>

<u>Qus.</u> Find $f'(x)$ where $f(x) = 2^{2^x}$.

<u>sol</u> Note that this is just $a^{g(x)}$ where $a = 2$ and $g(x) = 2^x$.

$$\Rightarrow \quad f'(x) = 2^{2^x} \cdot \ln 2 \cdot \frac{d}{dx}(2^x)$$

$$= 2^{2^x} \cdot \ln 2 \cdot 2^x \cdot \ln 2 = 2^{(2^x + x)} \cdot (\ln 2)^2$$

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