Lecture 2

Rational Expressions :-

A rational expression is a quotient of two polynomials $\frac{p(x)}{q(x)}$ with $q(x) \neq 0$.

e.g. $\frac{11}{x-3}$ is a rational expression as long as $x-3 \neq 0$ or $x \neq 3$.

$\frac{2x^2+5x+1}{x^3+5x^2+7x+4}$ is rational if denominator $\neq 0$.

Operations on rational expressions

1. Addition / Substraction

Recall how to add fractions

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

We add/substract rational expressions in the same way.

e.g.

$$\frac{x+1}{x+2} + \frac{5}{x+3} = \frac{(x+1)(x+3) + 5(x+2)}{(x+2)(x+3)}$$

$$= \frac{x^2+4x+3+5x+10}{x^2+5x+6}$$

$$= \frac{x^2+9x+13}{x^2+5x+6}$$

2. <u>multiplication</u> same as fractions; $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

e.g. $$\frac{x+1}{x+2} \cdot \frac{5}{x+3} = \frac{5x+1}{x^2+5x+6}$$

3. <u>Division</u> again, same as fractions. $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}$

So $$\dfrac{\frac{x+1}{x+2}}{\frac{5}{x+3}} = \frac{(x+1)(x+3)}{5(x+2)} = \frac{x^2+4x+3}{5x+10}.$$

Can we factor and simplify rational expressions just like we did the polynomials?

Yes! Just factor the numerator and the denominator separately and then cancel out common terms, if any.

e.g. $$\frac{x^2+7x+12}{x^2+2x-3} = \frac{\cancel{(x+3)}(x+4)}{\cancel{(x+3)}(x-1)} = \frac{x+4}{x-1}$$

Equations [Solving for x]

Observe that the quadratic formula is a way to solve quadratic equations, i.e., expressions of the form $Ax^2+Bx+C=0$. Similarly, we can solve for other type equations as well.

e.g.

1. Linear equations :- equations of the form $Ax+B=0$, i.e., the highest exponent of x is 1.

Solve : $3x+5=7$

$\Rightarrow 3x = 7-5 = 2 \Rightarrow x = \frac{2}{3}$

2. Quadratic Equations We already know how to solve them either by guessing the roots or by using the quadratic formula.

3. Rational equations Equations of the form $\frac{p(x)}{q(x)} = \frac{a(x)}{b(x)}$ with $q(x) \neq 0$ and $b(x) \neq 0$. Here, we will move all the terms towards the right hand side thus getting a single rational expression and then equate the numerator to 0.

e.g. Solve $\frac{x}{5} = \frac{2x}{x+2}$

$\Rightarrow \frac{x}{5} - \frac{2x}{x+2} = 0$ [moving everything to the RHS]

$\Rightarrow \frac{x(x+2) - 10x}{5(x+2)} = 0$

$\Rightarrow \frac{x^2 - 8x}{5(x+2)} = 0$ [getting a single rational expression]

$\Rightarrow x^2 - 8x = 0 \Rightarrow x(x-8) = 0$ [numerator $=0$]

$\Rightarrow \quad x = 0 \quad$ or $\quad x = 8.$

Inequalities $\quad (<, >, \leq, \geq)$

Note:- When solving an inequality remember that: Adding and substracting do not change the direction of the inequality. However, the sign is reversed when multiplying or dividing by a negative number.

e.g. Solve for x: $\quad 4 - 3x \leq 7 + 2x$

$\Rightarrow \quad 4 - 3x - 2x \leq 7 \qquad$ [no change ie $\leq$ sign]

$\Rightarrow \quad 4 - 5x \leq 7$

$\Rightarrow \quad -5x \leq 3$

$\Rightarrow \quad x \geq -\frac{3}{5} \qquad$ [$\leq$ reversed to $\geq$ as dividing by -5]

Thus $\quad x \geq -\frac{3}{5} \quad$ which we can also write as

- $x \in \left[-\frac{3}{5}, \infty\right)$ → round means the point is excluded. [Always round for $\pm\infty$]

  $\in$ "belongs to"

  [→ square means the point is included

- plot on a number line.

e.g. Solve for x: $-1 \leq 3x+4 < 7$

Let's solve both the inequalities at once!

$$-1 \leq 3x+4 < 7$$

$\Rightarrow \quad -1-4 \leq 3x < 7-4$ (substracting 4 from both sides)

$\Rightarrow \quad -5 \leq 3x < 3$

$\Rightarrow \quad -\frac{5}{3} \leq x < 1$ (no sign change as dividing by a positive number)

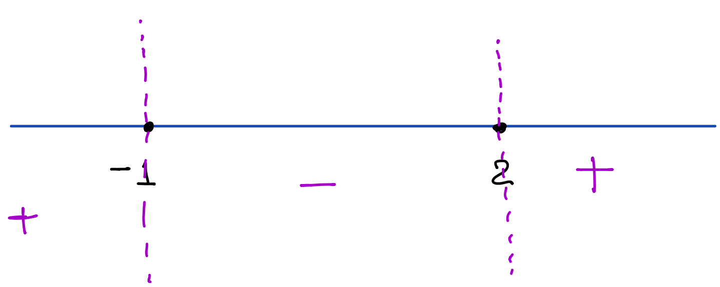
Thus $-\frac{5}{3} \leq x < 1$ or $x \in \left[-\frac{5}{3}, 1\right)$ or

e.g. Solve $x^2 - x - 2 \geq 0$.

We'll first factor the quadratic. So

$$(x-2)(x+1) \geq 0$$

Thus the expression above can be zero at $x = 2, -1$. So the test points are between the zeroes.



Both $x+1$ and $x-2$ are either both positive or both negative and hence their product is positive, wherever it shows +.

Thus the solution is $x \in (-\infty, -1] \cup [2, \infty)$

"or"