Lecture 14

Sometimes, even the strategies mentioned in the previ-
-ous lectures are not useful. In such cases, we will
have to use other results.

The Squeeze Theorem / Sandwich Theorem

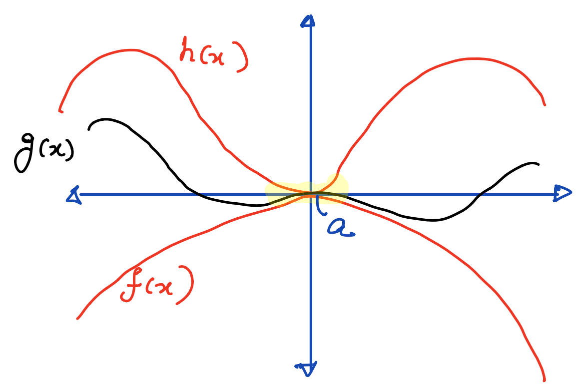
If $f(x)$, $g(x)$ and $h(x)$ are functions
such that

$$f(x) \le g(x) \le h(x)$$

around a and if

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

then $\lim_{x \to a} g(x) = L$.



As the graph indicates, we just need $f(x) \le g(x) \le h(x)$ around a.

Remark This theorem is very useful for calculating limits involving $\sin x$ and $\cos x$ as

$$-1 \le \sin x \le 1 \quad \text{and} \quad -1 \le \cos x \le 1.$$

e.g. Find $\lim_{x \to \infty} \frac{\sin x}{x^2}$.

Sol: Since $-1 \le \sin x \le 1 \Rightarrow \frac{-1}{x^2} \le \frac{\sin x}{x} \le \frac{1}{x^2}$

Thus in the statement of the squeeze theorem

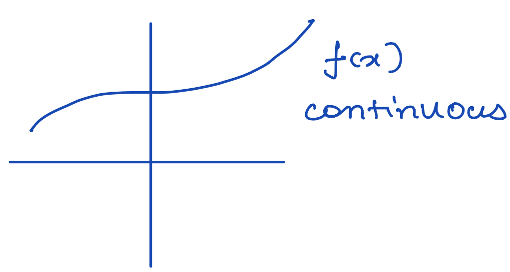
$f(x) = \frac{-1}{x^2}$, $g(x) = \frac{\sin x}{x^2}$ and $h(x) = \frac{1}{x^2}$

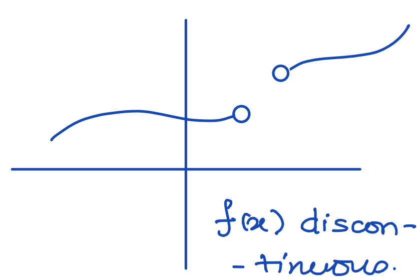
now $\lim_{x \to \infty} \frac{-1}{x^2} = 0 = \lim_{x \to \infty} \frac{1}{x^2}$

$\Rightarrow \lim_{x \to \infty} \left(\frac{\sin x}{x^2} \right) = 0.$

Continuity

Intuitively, a function is <u>continuous</u> if we can draw its graph without removing the pen from the paper.





Formally: A function $f(x)$ is continuous at $x = a$
if

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

We just write $\boxed{\lim_{x \to a} f(x) = f(a)}$

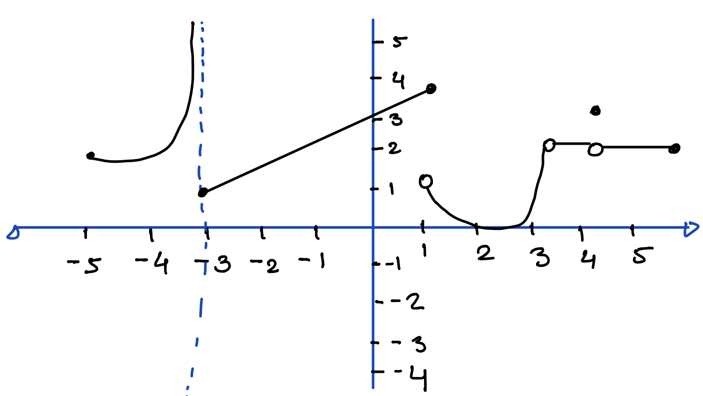
If I is an interval then we say that f is continuous on I if it is continuous at each point of I.

If $I = [a, b]$ then we mean $\lim_{x \to a^+} f(x) = f(a)$ and

$$\lim_{x \to b^-} f(x) = f(b)$$

E.g.:- polynomials, logarithms, exponential, $\sin x$ and $\cos x$ are all continuous functions.

Ex. Consider the graph of $f(x)$ given below.



We observe :-

$f(x)$ is discontinuous at

1) $x = -3$ (limit DNE)

2) $x = 1$ (limit DNE)

3) $x = 3$ ($f(3)$ DNE)

4) $x = 4$ ($\lim_{x \to 4} f(x)$ exists and $f(4)$ exists but $\lim_{x \to 4} f(x) = f(4)$).

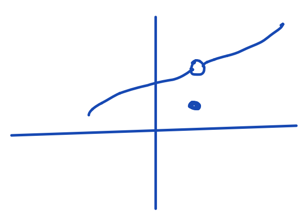
$f(x)$ is continuous everywhere else in $[-5, 5]$.

This example will help in understanding different types of discontinuity.

There are 3 types of discontinuity :-

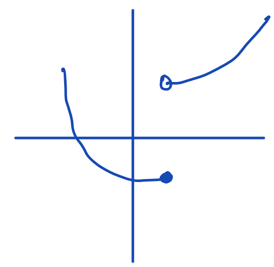
1. If $\lim_{x \to a} f(x)$ exists but $\lim_{x \to a} f(x) \neq f(a)$, it is a removable discontinuity.

(Thus $f(a)$ DNE or $\lim_{x \to a} f(x) \neq f(a)$).

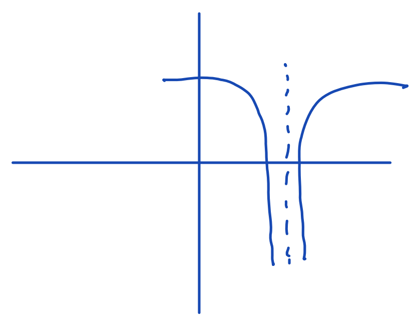


2. If both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are

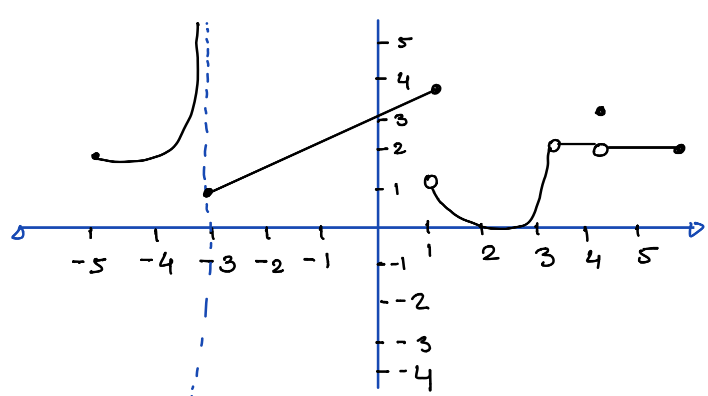
finite, BUT are not equal, then it is a jump discontinuity.



3. If $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$, then it is an infinite discontinuity.



Thus in the previous example, we have the following type of discontinuities:-



1) $\lim\limits_{x\to 3^-} f(x) = +\infty \Rightarrow$ infinite discontinuity at $x=-3$

2) $\lim\limits_{x\to 1^-} f(x) = 4$ and $\lim\limits_{x\to 1^+} f(x) = 1 \Rightarrow$ LHL $\neq$ RHL

$\Rightarrow$ jump discontinuity at $x=1$.

3) $f(3)$ DNE $\Rightarrow$ removable singularity at $x=3$.

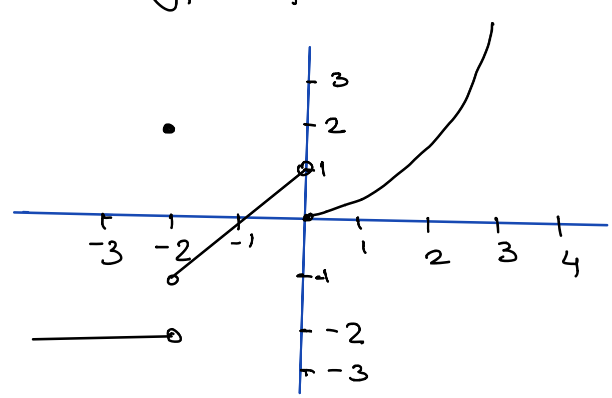
4) $\lim\limits_{x\to 4^-} f(x) = 2 = \lim\limits_{x\to 4^+} f(x)$ But $f(4) = 3$

Thus, removable singularity at $x=4$.

E.g. Sketch $f(x) = \begin{cases} -2 & \text{if } x < -2 \\ 2 & \text{if } x = -2 \\ x+1 & \text{if } -2 < x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

Where is $f(x)$ continuous? What types of discontinuities does $f(x)$ has?

Solution Graph of $f(x)$ is



- $\because \lim_{x \to -2^-} f(x) = -2$, $\lim_{x \to -2^+} f(x) = -1$ $\Rightarrow$ LHL and RHL exist but are not equal.

$\Rightarrow$ jump discontinuity at $x = -2$.

- $\lim_{x \to 0^-} f(x) = 1$ but $\lim_{x \to 0^+} f(x) = 0$

$\Rightarrow$ jump discontinuity at $x = 0$.

- $f(x)$ is continuous everywhere else.