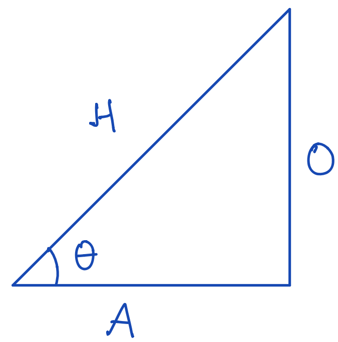
Lecture 11

Recall: SOH CAH TOA



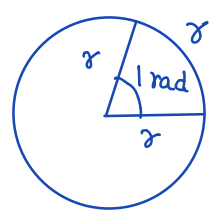
$\sin\theta = \dfrac{O}{H}$ $\qquad \cos\theta = \dfrac{A}{H}$

$\tan\theta = \dfrac{O}{A}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$

In calculus, angles are always measured in radians.

1 radian = angle that cuts off arc length equal to the radius of a circle



To convert degree ⟷ radians

$$\boxed{\text{degrees} = \frac{(\text{radians})\,180}{\pi} \qquad \text{radians} = \frac{(\text{degrees})\,\pi}{180}}$$

E.g. 1) $0^\circ = 0$ rad

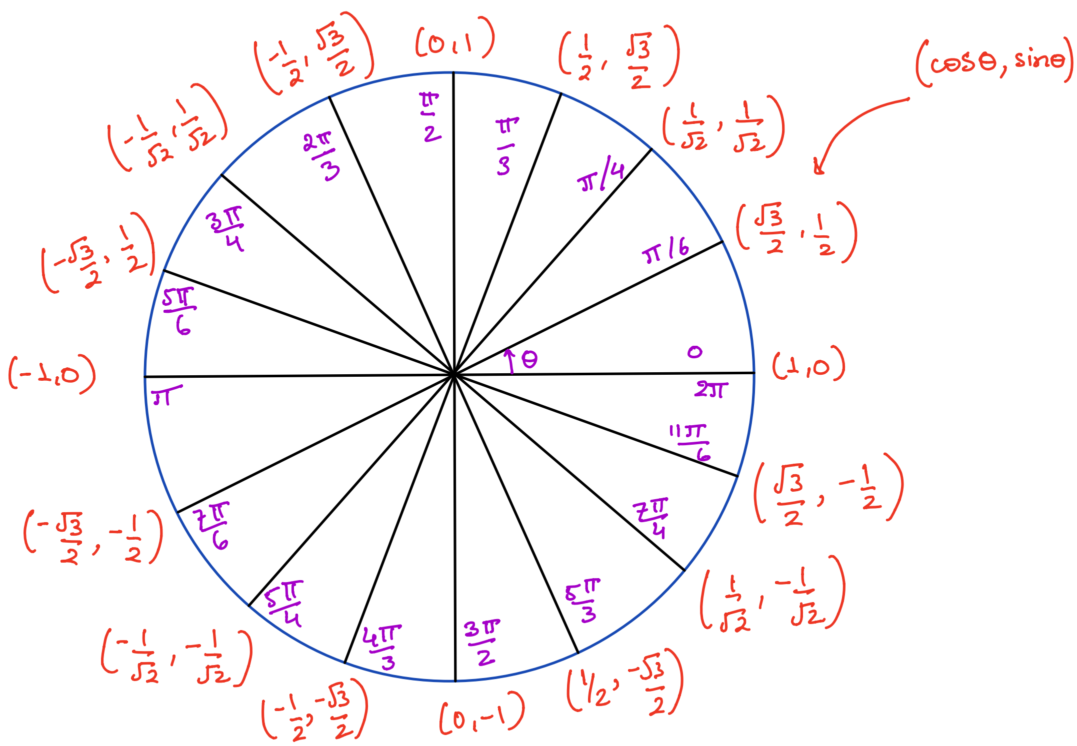
2) $30^\circ = \frac{30 \cdot \pi}{180} = \frac{\pi}{6}$ rad

3) $45^\circ = \frac{45 \cdot \pi}{180} = \frac{\pi}{4}$ rad

4) $60^\circ = \frac{60 \cdot \pi}{180} = \frac{\pi}{3}$ rad

5) $90^\circ = \frac{90 \cdot \pi}{180} = \frac{\pi}{2}$ rad and so on.

Most values of $\sin\theta$ and $\cos\theta$ are found using a calculator; but we should certain values by heart, which can be found on the unit circle.



For finding $\tan\theta$, note that $\tan\theta = \frac{O}{A} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{\sin\theta}{\cos\theta}$

Thus, $\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$

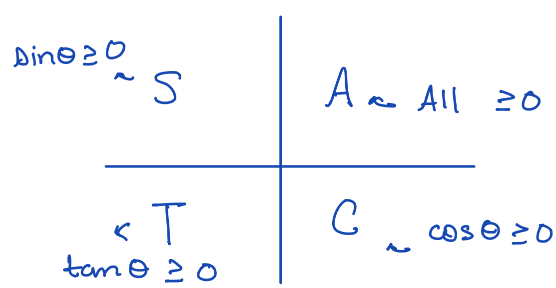
e.g.: $\tan\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

$\tan\left(\frac{\pi}{4}\right) = \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = 1$

$\tan(0) = \frac{0}{1} = 0$

$\tan\left(\frac{\pi}{2}\right) = \frac{1}{0}$ ⤳ not defined

As you can see from the unit circle that $\sin\theta$, $\cos\theta$, $\tan\theta$ etc takes both '+'ve and '−'ve values. To remember where they take what kind of values, we can use the CAST rule:-



Trigonometric Identities

There are lots of trig. identities. The most important one is

$$\boxed{\sin^2\theta + \cos^2\theta = 1} \qquad \text{——— } (1)$$

Divide (1) by $\sin^2\theta$ to get

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Divide (1) by $\cos^2\theta$ to get

$$\tan^2\theta + 1 = \sec^2\theta$$ and so on.

Sum/Difference of Angles

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

Double Angle

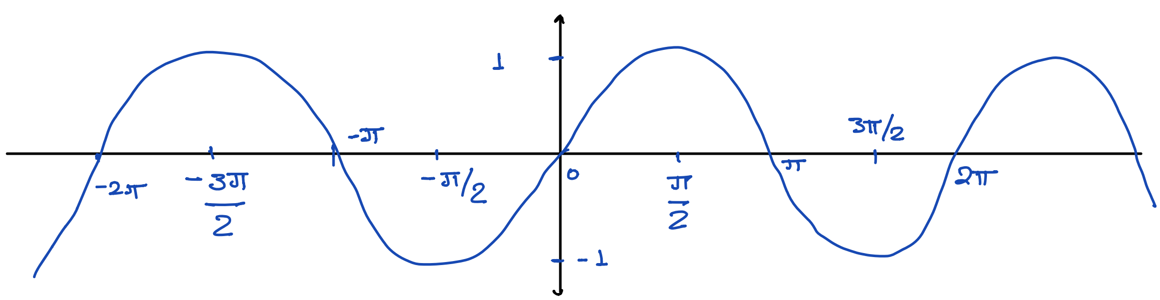
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

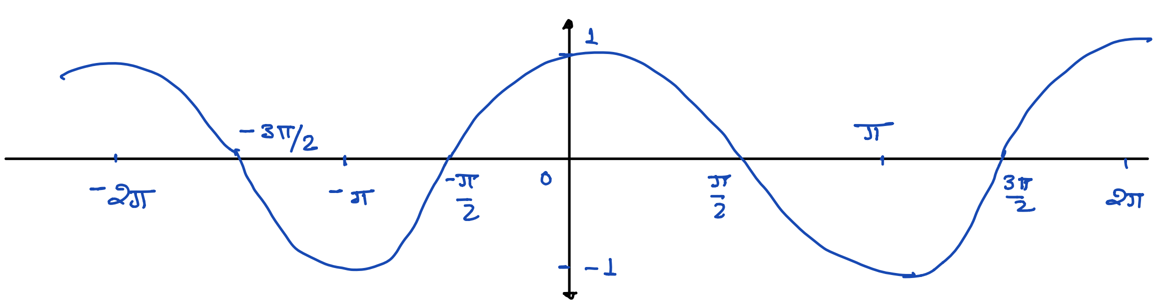
$$\therefore \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Graphs of Trig Functions

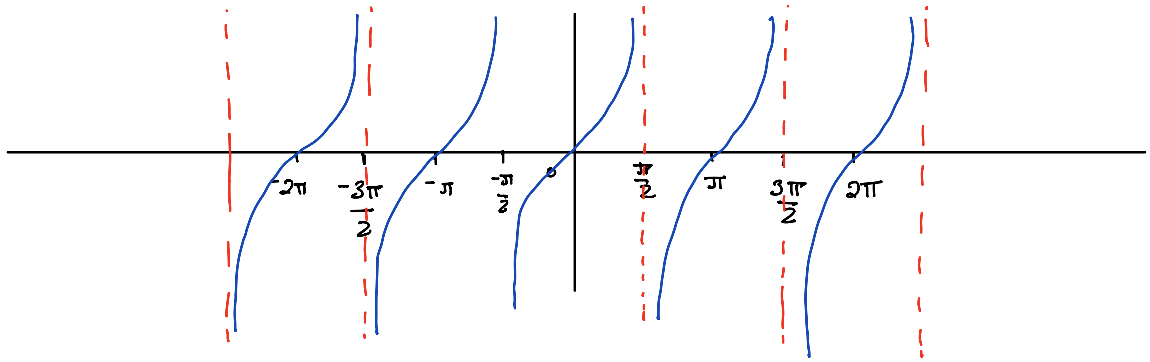
$y = \sin x$



$y = \cos x$



$y = \tan x$



As you can see that the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ starts repeating.

Defn A periodic function is a function $f(x)$ such that

$$f(x) = f(x+a)$$

for some positive number a.
'a' is called the period of f.

From the graphs we see that

- $\sin\theta$, $\cos\theta$ are periodic with period 2π
- $\tan\theta$ is periodic with period π.
  - $\cot\theta$ has period π.

Thus $\sec\theta$ and $\operatorname{cosec}\theta$ have period 2π

Also note from the graph that $\sin\theta$ and $\cos\theta$ reach the maximum height of 1 unit.

This is called the amplitude.

In general, if $y = A\sin(B(x-C)) + D$ or

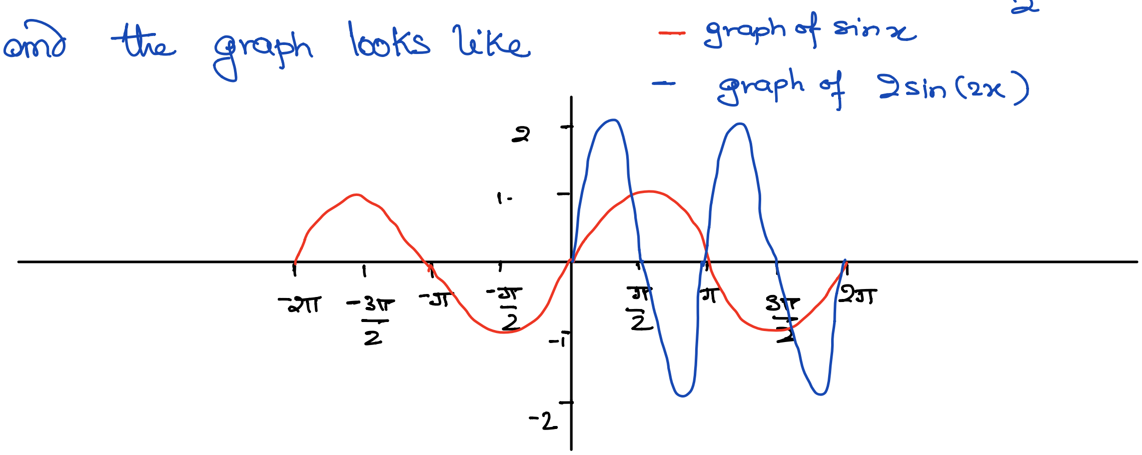
$y = A\cos(B(x-C)) + D$ then

A = amplitude Period = $\frac{2\pi}{B}$

C = horizontal shift D = vertical shift

e.g. $y = 2\sin(2x)$ has amplitude 2, period $= \frac{2\pi}{2} = \pi$

and the graph looks like



Solving Trig Equations

e.g. Solve for θ : 1) $\sin 2\theta = \cos\theta$

2) $4\cos\theta = 4 + \sin^2\theta$

Sol:- 1) note, $\sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow \quad 2\sin\theta\cos\theta = \cos\theta \quad \Rightarrow \quad \cos\theta(2\sin\theta - 1) = 0$$

$$\Rightarrow \quad \cos\theta = 0 \quad \text{or} \quad 2\sin\theta - 1 = 0, \text{ i.e., } \sin\theta = \frac{1}{2}$$

So, $\cos\theta = 0 \Rightarrow \theta = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \ldots$

Thus the solution is

$$\boxed{\theta = \begin{cases} \frac{\pi}{2} + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \\ \frac{3\pi}{2} + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \\ \frac{\pi}{6} + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \\ \frac{5\pi}{6} + 2k\pi & (k = 0, \pm 1, \pm 2, \ldots) \end{cases}}$$

So, the point to remember is that find all solutions θ in $[0, 2\pi]$ and then take care of the repeats from periodicity.

2) $4\cos\theta = 4 + \sin^2\theta$

$$= 4 + (1-\cos^2\theta) \qquad (\text{Using } \sin^2\theta + \cos^2\theta = 1)$$

$$= 5 - \cos^2\theta$$

$$\Rightarrow \quad \cos^2\theta + 4\cos\theta - 5 = 0$$

$$\Rightarrow \quad (\cos\theta + 5)(\cos\theta - 1) = 0 \quad \Rightarrow \quad \cos\theta = -5$$

[not possible as $-1 \le \cos\theta \le 1$]

or $\cos\theta = 1$

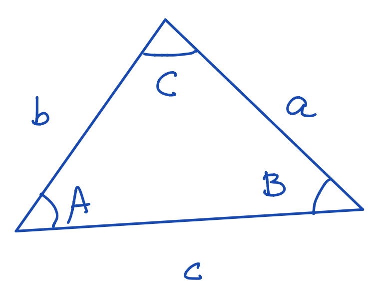
Thus $\cos\theta = 1$

$$\Rightarrow \quad \theta = 0, 2\pi, 4\pi, 6\pi, \ldots$$

$$\boxed{\theta = 2k\pi} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Sine and Cosine Laws

If we have any triangle

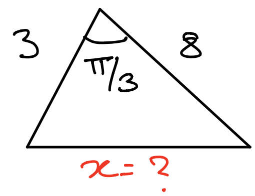


then

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$ — sine Law

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos(C)}$$ — cosine Law

e.g. solve for the unknowns :-

1) 

Use cosine laws to get

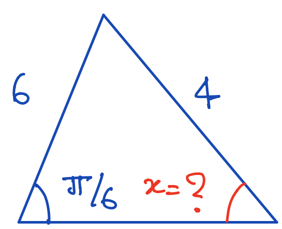
$$x^2 = 3^2 + 8^2 - 2(3)(8)\cos\frac{\pi}{3}$$

$$= 9 + 64 - 48 \cdot \frac{1}{2}$$

$$= 73 - 24 = 49$$

$\Rightarrow$ $x = \pm 7$ But a length cannot be negative

$\Rightarrow$ $\boxed{x = 7}$

2) 

Sine law $\Rightarrow$

$$\frac{\sin(\pi/6)}{4} = \frac{\sin x}{6}$$

$\Rightarrow$ $\frac{1}{2} \cdot 6 = 4 \sin x$

$\Rightarrow$ $\sin x = \frac{3}{4}$

$\Rightarrow$ $x = \sin^{-1}\left(\frac{3}{4}\right)$ (can find using a calculator).

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