Seminar on special holonomy manifolds
Universität Hamburg, Winter Semester 2025

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Talks: Tuesdays 10:15 - 11:45 in Sed 19, Room 203

Language: English

Short Description

The main topic of this seminar is to study *holonomy group* of a Riemannian metric g on a Riemannian manifold M^n, to study what type of information does the holonomy group of g $Hol(g)$ convey about the metric and the manifold. The holonomy group, simply speaking, is the group "generated by parallel transport along loops". In the case of Riemannian manifolds, there is a very nice list of possible Lie groups which can occur as holonomy groups of the metric. This list was compiled and proved by Berger [Ber55] which says that if apart from certain cases (we will learn about what those cases are), we have the following table:

Dimension	Holonomy group	Remarks
n	SO(n)	Generic Riemannian manifold
2m	U(m)	Kähler
2m	SU(m)	Calabi-Yau
4q	Sp(q)	Hyper-Kähler
4q	$\mathrm{Sp(q)} \cdot \mathrm{Sp(1)}$	Quaternionic-Kähler
7	G_2	G_2-holonomy
8	Spin(7)	Spin(7)-holonomy

The objective of the first half of the seminar will be to understand the concepts involved in Berger's theorem and to understand a proof of the theorem itself. The second half of the seminar will focus on G_2 and Spin(7)-holonomy metrics and if time permits, understanding *calibrated geometries* of these manifolds.

A rough outline of the talks during the seminar is as follows.

(1) Review of Riemannian Geometry: Discuss Riemannian metric, Levi-Civita connection, curvature tensor, parallel transport [Joy00].
(2) Define the holonomy group $Hol(g)$ of a Riemannian manifold and discussing that it a Lie group [Joy00].
(3) Understand the holonomy principle - reduction of the holonomy gives rise to parallel tensors [Joy00].
(4) Discuss Riemannian symmetric spaces and their properties [Joy00].
(5) Discussion of the Riemann curvature tensor as a self-adjoint operator on 2-forms and discuss the Ambrose–Singer Theorem, relating $Hol(g)$ and the Riemann curvature tensor [Joy00].
(6) State Berger's Classification of Riemannian Holonomy Groups and discuss a sketch of Berger's proof [Joy00].
(7) Start with the exceptional Lie group G_2, its description of being the automorphism group of the Octonions $\mathbb{O}$. Discuss how a G_2-structure is equivalent to the existence of a non-degenerate 3-form φ on the manifold. Describe how a G_2-structure induces a Riemannian metric and an orientation [Joy00, Kar09, SW17, Sal89].

(8) Understand the decomposition of differential forms as per the irreducible G_2-representations, the *intrinsic torsion forms* of a G_2-structure, the 16-classes of G_2-structures and remarks about them [Joy00, Kar09, Sal89].
(9) State and prove the G_2-Bianchi identity, Fernández-Gray's theorem on torsion-free G_2-structures being closed and co-closed, Bonan's result on torsion-free G_2-structures having Ricci-flat underlying metric [Kar09, Sal89].
(10) Explain Bryant–Salamon's construction of the first examples of complete non-compact G_2–manifolds [BS89].
(11) Discuss Joyce's results on the condition for the existence of a torsion-free G_2-structure in the cohomology of a closed G_2-structure being equivalent to solving a non-linear elliptic PDE [Joy00].
(12) State and prove Joyce's theorem on the moduli space of torsion-free G_2-structures [Joy00].
(13) State Joyce's fundamental perturbation theorem: "almost" torsion-free G_2-structures can be perturbed to a torsion-free one [Joy00].
(14) Discuss Joyce's generalised Kummer construction of the first examples of compact manifolds with holonomy G_2 [Joy00].
(15) Introduce calibration forms and prove calibrated submanifolds being absolutely volume minimizing in their homology class [HL82].
(16) Discuss calibrated submanifolds of G_2-manifolds and their properties [HL82].

References

[Ber55] Marcel Berger, *Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes*, Bull. Soc. Math. Fr. **83** (1955), 279–330 (French). ↑1

[BS89] Robert L. Bryant and Simon M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), no. 3, 829–850 (English). ↑2

[HL82] Reese Harvey and H. Blaine Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157 (English). ↑2

[Joy00] Dominic D. Joyce, *Compact manifolds with special holonomy*, Oxford Math. Monogr., Oxford: Oxford University Press, 2000 (English). ↑1, 2

[Kar09] Spiro Karigiannis, *Flows of G_2-structures. I.*, Q. J. Math. **60** (2009), no. 4, 487–522 (English). ↑1, 2

[Sal89] Simon Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes Math. Ser., vol. 201, Harlow: Longman Scientific &| Technical; New York: John Wiley &| Sons, 1989 (English). ↑1, 2

[SW17] Dietmar A. Salamon and Thomas Walpuski, *Notes on the octonions*, Proceedings of the 23rd gökova geometry-topology conference, gökova, turkey, may 30 – june 3, 2016, 2017, pp. 1–85 (English). ↑1