Extrinsic Geometric Flows
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Problem Set 7
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Problems

(1) Calculate the second fundamental form and the mean curvature of the S^n_r of radius r with the round metric when viewed as a submanifold of $\mathbb{R}^{n+1}$.

(2) Let $X : M^n \to \mathbb{R}^{n+1}$ be a compact immersed hypersurface. Prove that

$$\mathrm{Area}(M) = -\frac{1}{n}\int_M \langle X, \vec{H}\rangle d\mu.$$

(3) Consider an embedded submanifold $X : M^n \to \mathbb{R}^{n+1}$.

(a) If $p \in M$ is an umbilic point, i.e., $A_p = \alpha_p g_p$, prove that $\alpha_p = \frac{H(p)}{n}$.

(b) Suppose that M is totally umbilic. Prove that $\nabla A = 0$.

(c) Deduce from the above that connected, totally umbilic hypersurfaces of $\mathbb{R}^{n+1}$ are open subsets of hyperplanes or round spheres.

(4) Use the Simons's inequality to prove the following result due to Choi–Schoen[1]

There exist $\epsilon, \rho > 0$ such that if $r_0 < \rho$, $M^2 \subset \mathbb{R}^3$ is a compact minimal surface (so $H = 0$) with $\partial M \subset \partial B_{r_0}(x), 0 < \delta \leq 1$ and

$$\int_{B_{r_0}\cap M} |A|^2 < \delta\epsilon,$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0-\sigma}(x)$,

$$\sigma^2|A|^2(y) \leq \delta.$$

Remark: Try to prove this by a contradiction argument. You might need to use the mean value inequality for minimal surfaces: Suppose $M^2 \subset \mathbb{R}^3$ is minimal with $x_0 \in M^2$ and $s > 0$ satisfying $B_s(x_0) \cap \partial M^2 = \emptyset$. If f is a nonnegative function on M^2 satisfying $\Delta f \geq -\lambda s^{-2} f$ then

$$f(x_0) \leq e^{\frac{\lambda}{2}} \frac{\int_{B_s(x_0)\cap M^2} f}{\mathrm{Vol}(B_s \subset \mathbb{R}^3)}.$$

What should be the f in above situation?

[1] H.I. Choi and R. Schoen, *The space of minimal embeddings of a surface into a three manifold of positive Ricci curvature*, Invent. Math. 81 (1985) 387-394.