Problem Set 6
Due date: 05.07.2023

Problems

(1) Along the CSF on $[0, T)$, prove that we have an estimate of the form

$$\kappa^2 \leq \left(\tfrac{2\pi}{L}\right)^2 (1 + C(T-t))$$
$$L \leq 2\pi\sqrt{2(T-t)}(1 + C(T-t))$$

for some constant C.

(2) Let $X(t), t \in [0, T)$ be a CSF and consider $\Phi(X, \tau) = (4\pi\tau)^{-\frac{1}{2}} e^{-\frac{|X|^2}{4\tau}}$ be the backwards heat kernel with $\tau = T - t$. Prove that

$$\partial_t \Phi + \partial_s^2 \Phi = \left(\kappa^2 - \left(\kappa - (2\pi)^{-1}\langle X, N\rangle\right)^2\right)\Phi. \tag{0.1}$$

Use the above to prove that

$$\frac{d}{dt}\int_{M^1} \Phi(X, \tau) ds = -\int_{M^1} \left(\kappa - \frac{\langle X, N\rangle}{2\tau}\right)^2 \Phi(X, \tau) ds.$$

This is the CSF version of the Huisken's monotonicity formula. We'll see the general version for MCF in the lectures.

(3) Let $M^n = \text{graph}f = \{(x, f(x)) \mid x \in \Omega\}$ with Ω a domain in $\mathbb{R}^n$ and $f \in C^\infty(\Omega)$. A graph hypersurface $M^n = \text{graph}f$ in $\mathbb{R}^{n+1}$ is given by the embedding $X : M^n \to \mathbb{R}^{n+1}$

$$X(x) = (x, f(x)).$$

Find the unit normal vector field to M and using that, calculate the expression for the second fundamental form of M and the mean curvature of M. (You can use the fact that for hypersurfaces, the mean curvature H=trace of the second fundamental form using the metric.)

(4) Suppose that the variation of the Riemannian metric is given by a symmetric 2-tensor h, i.e., suppose that $\frac{\partial g(t)}{\partial t} = h$. Find the variation of the Christoffel symbols Γ^i_{jk} and the volume form of the metric. **Remark:** your expression should be in terms of h and keep in mind that a connection is NOT a tensor because of the Leibniz rule but a difference of two connections is a tensor.