Problem Set 5
Due date: 14.06.2023

Problems

(1) Let $X(t)$ be a convex solution to the CSF. Recall the normal angle $\theta(t) : M^1 \to \mathbb{R}/2\pi\mathbb{Z}$ for every t. Then $\kappa \circ \theta^{-1}$ can be considered as a function on $\mathbb{R}/2\pi\mathbb{Z} \times [0, T)$. Suppose $\frac{\partial}{\partial t}$ denotes the time-derivative in original coordinates (u, t) and $\frac{\partial}{\partial \tau}$ denotes the time-derivative in (θ, t) coordinate. Prove that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \kappa^{-1}\kappa_s \frac{\partial}{\partial s}.$$

(2) In the same situation as Prob. 1 and viewing the curvature κ as a function of (θ, t), find the expression for κ_τ.

(3) Prove that if $X_c(u, t) = cX(u, c^{-2}t)$ is a resclaed solution to the CSF then $\kappa_c(\tau) = c^{-1}\kappa(c^{-2}\tau)$ is the soution to the equation you derive in Prob. 2. Note that the rescaling is a "parabolic rescaling", i.e., time scales like distance^2.

(4) Give a convex solution to the CSF, consider the quantity

$$Q = \kappa(\kappa_{\theta\theta} + \kappa).$$

Prove that $Q = 0$ on the Grim reaper and prove that $Q_\tau = \kappa^2 Q_{\theta\theta} + 2\kappa\kappa_\theta Q_\theta + 2Q^2$.

(5) Usng the maximum principle, prove that on any immersed, convex solution $X : S^1 \times [\alpha, T) \to \mathbb{R}^2$

$$Q + \frac{1}{2(t - \alpha)} \geq 0$$

for $t \in [\alpha, T)$. This is the *Harnack inequality* for convex solutions to CSF.