Extrinsic Geometric Flows
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Problem Set 2
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Problems

(1) For $u : T^n \times [0,T) \to \mathbb{R}$, a positive solution to the heat equation, we define the **Nash entropy** of u by
$$N(u) = -\int_M u \log u d\mu.$$
Prove that the entropy is nondecreasing in time.

(2) Let u be a solution to the heat equation and let ϕ be a solution to the L^2-*adjoint heat equation*, i.e.,
$$(\partial_t + \Delta\phi) = 0.$$
One can define the *weighted volume* of the space as $\phi d\mu$. Prove that
(a) the "mass" of u, $\int_{T^n} u\phi d\mu$ is constant with respect to the weighted volume.
(b) Prove that the L^2-norm of u with respect to the weighted volume is strictly decreasing unless u is a constant function.

(3) We prove some results about smooth curves.
(a) Prove Wirtinger's inequality: Let $f : [0,\pi] \to \mathbb{R}$ be a smooth function with $f(0) = f(\pi) = 0$. Then
$$\int_0^\pi (f_t)^2 dt \geq \int_0^\pi f^2(t) dt$$
with equality if and only if $f(t) = C \sin t$ with C being a constant.
(b) Using the Wirtinger's inequality, prove the following result know as the **Isoperimetric inequality.**: Let γ be a simple closed curve and let $L(\gamma)$ be the length of γ and $A(\gamma)$ be the area enclosed by γ. Prove that
$$L^2(\gamma) \geq 4\pi A(\gamma)$$
with equality if and only if γ is a circle.

(4) For $k \in \mathbb{N}$ and $r \in (0,1)$, consider the curve $X_{k,r}(u) = \left(-\cos u + kr\cos\frac{u}{k}, \sin u - kr\sin\frac{u}{k}\right)$. Compute the curvature of $X_{k,r}$.