Problem Set 1
Due date: 03.05.2023

Problems

(1) Prove that
$$\int_{\mathbb{R}^n} \rho(x,t)d\mu = 1$$
where ρ is the fundamental solution to the heat equation on $\mathbb{R}^n$. The reason ρ is important is because of the following: prove that
$$\lim_{t\to 0}\int_{\mathbb{R}^n} \rho(x,t)\varphi(x)d\mu = \varphi(0)$$
for all smooth functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ with compact support.

(2) Let u be a solution to the heat equation. We say that u is invariant under **Galilean boosts** if
$$u_\varepsilon(x,t) := e^{-(x-t\varepsilon V)\cdot\varepsilon V} u\,(x - 2t\varepsilon V, t)$$
with $V \in \mathbb{R}^n$, is again a solution to the heat equation. Prove that the fundamental solution ρ to the heat equation on $\mathbb{R}^n$ is invariant under all Galilean boosts. In fact, find all solutions $u : \mathbb{R}\times(0,\infty)$ of the heat equation on $\mathbb{R}$ which are invariant under the Galilean boosts.

(3) Let $p > 1$. Prove that the negative gradient flow of the energy density
$$e(u) = \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}$$
for $u : \mathbb{R}^n \times [0,T)$ is the equation
$$(\partial_t - \Delta)u = |u|^{p-1}u.$$
This is an example of a *semi-linear heat equation.*

(4) Let (M^n, g) be a compact Riemannian manifold satisfying $\mathrm{Ric} \geq 0$ with Ric being the Ricci curvature of g and let $u : M \times [0,T) \to \mathbb{R}$ be a solution to the heat equation. Show that
$$(\partial_t - \Delta)|\nabla u|^2 \leq -2|\nabla^2 u|^2.$$
Deduce a Bernstein-type decay estimate for $|\nabla u|^2$.