Let us see some *non-examples* of smooth embeddings. Below we give some examples where one or the other criteria for being a smooth embedding fails.

Example 7.18 (A Smooth Topological Embedding)**.** The map $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3, 0)$ is a smooth map and a topological embedding, but it is not a smooth embedding. The map γ is smooth and is a topological embedding as

$$\gamma(\mathbb{R}) = \{(x, 0) : x \in \mathbb{R}\},$$

which is the x-axis in $\mathbb{R}^2$. The inverse map from the image $\gamma(\mathbb{R})$ to $\mathbb{R}$:

$$\gamma^{-1}(x, 0) = \sqrt[3]{x}.$$

Since $x \mapsto \sqrt[3]{x}$ is continuous, γ^{-1} is continuous. Therefore γ is a homeomorphism of $\mathbb{R}$ onto its image, so it is a topological embedding.

But γ is not a smooth embedding as

$$\gamma'(t) = (3t^2, 0).$$

and hence at $t = 0$,

$$\gamma'(0) = (0, 0).$$

Thus the differential

$$d\gamma_0 : T_0\mathbb{R} \to T_{(0,0)}\mathbb{R}^2$$

is the zero map, so it is not injective.

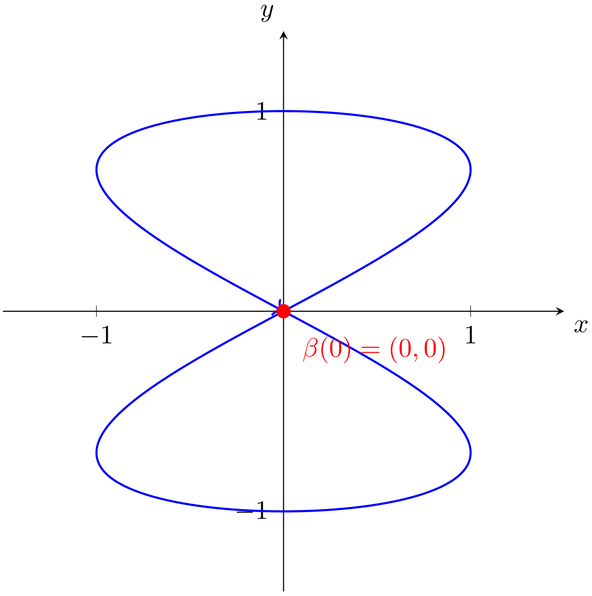
Example 7.19 (The Figure-Eight Curve)**.** Consider the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ defined by

$$\beta(t) = (\sin 2t,\ \sin t).$$

Its image looks like a figure-eight in the plane, see the figure below. We show that β is *not* a smooth embedding. We have

$$\beta'(t) = (2\cos 2t, \cos t),$$

and hence$\beta'(t) \neq 0$ for all $t \in (-\pi, \pi)$ as, at $t = \frac{\pi}{2}$, $\beta'(\frac{\pi}{2}) = \big(2\cos\pi, \cos\frac{\pi}{2}\big) = (-2, 0) \neq 0$. and at $t = 0$, $\beta'(0) = (2, 1) \neq 0$.



We check that β is *not* a homeomorphism onto its image. The inverse $\beta^{-1} : \beta((-\pi, \pi)) \to (-\pi, \pi)$ is **not continuous**. First observe that

$$\beta(0) = (\sin 0, \sin 0) = (0, 0).$$

Now define the sequence

$$t_n = \pi - \frac{1}{n} \in (-\pi, \pi), \qquad n \geq 2.$$

Then $t_n \to \pi$ as $n \to \infty$, and

$$\begin{aligned} \sin 2t_n &= \sin\bigl(2\pi - \tfrac{2}{n}\bigr) = -\sin \tfrac{2}{n} \to 0, \\ \sin t_n &= \sin\bigl(\pi - \tfrac{1}{n}\bigr) = \sin \tfrac{1}{n} \to 0. \end{aligned}$$

Therefore

$$\beta(t_n) = \bigl(-\sin \tfrac{2}{n}, \sin \tfrac{1}{n}\bigr) \longrightarrow (0,0) = \beta(0) \qquad \text{in } \mathbb{R}^2.$$

So $\beta(t_n) \to \beta(0)$ in the subspace topology on $\beta((-\pi, \pi))$.

If β^{-1} were continuous, we would need

$$\beta^{-1}(\beta(t_n)) = t_n \longrightarrow \beta^{-1}(0,0) = 0.$$

But $t_n \to \pi \neq 0$. Therefore β^{-1} is not continuous at $(0,0)$, so β is not a homeomorphism onto its image.

Example 7.20. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus, and let α be any irrational number. The map $\gamma : \mathbb{R} \to \mathbb{T}^2$ given by

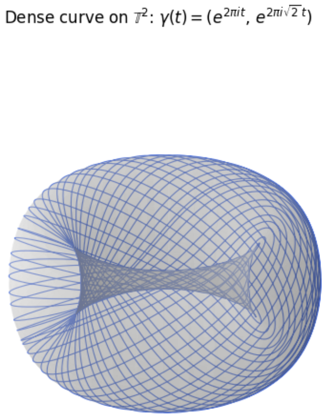
$$\gamma(t) = \left(e^{2\pi i t},\ e^{2\pi i \alpha t}\right),$$

is a smooth immersion because $\gamma'(t)$ never vanishes. It is also injective, because $\gamma(t_1) = \gamma(t_2)$ implies that both $t_1 - t_2$ and $\alpha t_1 - \alpha t_2$ are integers, which is impossible unless $t_1 = t_2$. However, γ is *not* a homeomorphism onto its image. To see this, let $\varepsilon > 0$. Then we can find $n, m \in bZ$ such that $|\alpha n - m| < \varepsilon$. Thus,

$$|\gamma(n) - \gamma(0)| = |(e^{2\pi i n}, e^{2\pi i \alpha n}) - (1,1)| = |(1, e^{2\pi i \alpha n}) - (1,1)| \leq |e^{2\pi i \alpha n} - e^{2\pi i m}| \leq |2\pi(\alpha n - m)| \leq 2\pi\varepsilon.$$

Thus, $\gamma(0) \in \overline{\gamma(\mathbb{Z})}$ but $\mathbb{Z}$ has no limit point in $\mathbb{R}$.

Of course, if $\alpha \in \mathbb{Q}$ then γ is not even injective.


Dense curve on $\mathbb{T}^2$: $\gamma(t) = (e^{2\pi i t}, e^{2\pi i\sqrt{2}\, t})$

7.4 Submanifolds

As the name suggests, submanifolds should be part of a manifold which should be manifolds in their own right. This is the nottion we want to make precise in this section. We will give two ways to define a submanifold. First, let us recall from linear algebra that a linear subspace of a vector space, for instance

$$\mathbb{R}^\ell \times \{0\} = \{(x^1, \ldots, x^\ell, 0, \ldots, 0) \in \mathbb{R}^n \mid (x^1, \ldots, x^\ell) \in \mathbb{R}^\ell\} \subset \mathbb{R}^n. \tag{7.6}$$

is definitely a manifold in its own right and in fact, this is the typical example of a submanifold. Indeed, any ℓ-dimensional subspace of an n-dimensional vector space looks like (7.6) after a suitable linear change of coordinates. The notion of a smooth submanifold generalizes this idea. We first start with the following definition.

Definition 7.21 (Slice charts)**.** A chart $(\mathcal{U}, x)$ on an n-manifold M is called an **ℓ-dimensional slice chart** for a subset $L \subset M$ if

$$L \cap \mathcal{U} = x^{-1}(\mathbb{R}^\ell \times \{0\}),$$

i.e. the points in $\mathcal{U}$ belong to L if and only if their coordinates $x^{\ell+1}, \ldots, x^n$ vanish.

Definition 7.22. Suppose M is a smooth n-manifold. A subset $L \subset M$ is called an **ℓ-dimensional smooth submanifold** of M if M admits a collection of smooth slice charts for L whose domains cover every point of L.

Example 7.23. If we use the polar coordinates for describing the smooth structure on $S^1 \subset \mathbb{R}^2$ with the two charts (U, θ) and (V, ϕ), where $\theta(U) = (0, 2\pi)$ and $\phi(V) = (-\pi, \pi)$., then these two coordinates were defined on open subsets of S^1, but they also have natural extensions to open subsets of $\mathbb{R}^2$, namely

$$\mathcal{U}' := \{tv \in \mathbb{R}^2 \mid v \in U,\ t > 0\}, \qquad \mathcal{V}' := \{tv \in \mathbb{R}^2 \mid v \in V,\ t > 0\}.$$

Recall that a **1-dimensional slice chart** for $S^1 \subset \mathbb{R}^2$ is a chart $(\mathcal{U}', (x^1, x^2))$ on $\mathbb{R}^2$ such that a point of $\mathcal{U}'$ lies on S^1 if and only if $x^2 = 0$.

Define the radial and angular coordinates on $\mathbb{R}^2 \setminus \{0\}$ as usual, and set:

$$\rho := r - 1 \quad \text{so that} \quad S^1 = \{r = 1\} = \{\rho = 0\}.$$

Define

$$\mathcal{U}' = \mathbb{R}^2 \setminus \big(\{0\} \cup \{(x, 0) \mid x > 0\}\big).$$

which is $\mathbb{R}^2$ with the origin and the positive x-axis removed. The coordinate map is:

$$(\theta,\ \rho) : \mathcal{U}' \to (0, 2\pi) \times (-1, \infty),$$

where $\theta \in (0, 2\pi)$ is the polar angle and $\rho = \sqrt{x^2 + y^2} - 1$. Then:

$$S^1 \cap \mathcal{U}' = (\theta, \rho)^{-1}(\mathbb{R} \times \{0\}),$$

since $\rho = 0 \iff r = 1$. Similarly,

$$\mathcal{V}' = \mathbb{R}^2 \setminus \big(\{0\} \cup \{(x, 0) \mid x < 0\}\big).$$

which is $\mathbb{R}^2$ with the origin and the **negative** x-axis removed. The coordinate map is

$$(\phi,\ \rho) : \mathcal{V}' \to (-\pi, \pi) \times (-1, \infty),$$

where $\phi \in (-\pi, \pi)$ is the polar angle and $\rho = \sqrt{x^2 + y^2} - 1$. Then:

$$S^1 \cap \mathcal{V}' = (\phi, \rho)^{-1}(\mathbb{R} \times \{0\}).$$

So $S^1 \subset \mathcal{U}' \cup \mathcal{V}'$, and the two slice charts cover every point of S^1, confirming that S^1 is a **smooth 1-dimensional submanifold** of $\mathbb{R}^2$.

Let us now prove that a submanifold of a manifold is also a manifold in its own right.

Proposition 7.24. *If L is an ℓ-dimensional C^k-submanifold of an n-dimensional C^k-manifold M, then L inherits naturally from M the structure of an ℓ-dimensional C^k-manifold such that the inclusion map $i : L \hookrightarrow M$ is of class C^k. Moreover, for each $p \in L$, the tangent space T_pL is naturally an ℓ-dimensional linear subspace of T_pM.*

Proof. We associate to every slice chart $(\mathcal{U}, x)$ for $L \subset M$ a chart of the form $(\mathcal{U} \cap L, x_L)$ on L, where we use the coordinate projection

$$\pi_\ell(x^1, \ldots, x^n) := (x^1, \ldots, x^\ell)$$

to define

$$x_L = \pi_\ell \circ x|_{\mathcal{U} \cap L} : \mathcal{U} \cap L \to \mathbb{R}^\ell.$$

Since L is a submanifold, L can be covered by slice charts and hence the collection of all charts of this form defines an atlas on L. We now check the compatibility conditions. Given two such charts $(\mathcal{U} \cap L, x_L)$ and $(\mathcal{V} \cap L, y_L)$ derived from two C^k-compatible slice charts $(\mathcal{U}, x)$ and $(\mathcal{V}, y)$, the transition map $y \circ x^{-1}$ preserves the subspace $\mathbb{R}^\ell \times \{0\} \subset \mathbb{R}^n$, and its restriction to the intersection of its domain with this subspace is the transition map $y_L \circ x_L^{-1}$, which is therefore of class C^k as $y \circ x^{-1}$ is of class C^k. Similarly, second countability and Hausdorff property for L

follows from that of M and thus L is a C^k-manifold. The inclusion map $i : L \hookrightarrow M$, with respect to any slice chart $(\mathcal{U}, x)$ and the associated chart $(\mathcal{U} \cap L, x_L)$ on L, looks like

$$(x^1, \ldots, x^\ell) \mapsto (x^1, \ldots, x^\ell, 0, \ldots, 0),$$

which is clearly smooth, thus the inclusion is of class C^k.[2]

For each $p \in L$, the tangent map $di_p : T_pL \to T_pM$ is simply the canonical inclusion $T_pL \hookrightarrow T_pM$ defined by regarding each path in L as a path in M. Since its image is a linear subspace, it gives a canonical isomorphism of T_pL to a linear subspace of T_pM. ☐

Thus a submanifold $L \subset M$ is a manifold and from now on, we will assume that L is endowed with the differentiable structure described in Proposition 7.24.

Exercise 7.25. Suppose M and N are both C^k-manifolds and $f : M \to N$ is a map of class C^k. Prove:

(a) For any C^k-submanifold $L \subset M$, the restriction $f|_L : L \to N$ is also a map of class C^k.

(b) If $L \subset N$ is a C^k-submanifold such that $f(M) \subset L$, then the resulting map $f : M \to L$ is also of class C^k.

7.5 Embeddings and level sets

We nowtry to understand the relation between embeddings and submanifolds which will also allow us to produce many more examples of submanifolds. The main result in this direction is the following.

Theorem 7.26. *If $f : M \to N$ is an embedding, then its image $f(M)$ is a smooth submanifold of N.*

Proof. We want to produce slice charts for $f(M) \subset N$. Suppose $q \in f(M)$. Since f is injective, there is a unique point $p \in M$ such that $f(p) = q$, and since f is an immersion, the Rank Theorem 7.11 gives us charts $(\mathcal{U}, x)$ on M and $(\mathcal{V}, y)$ on N with $x(p) = 0$ and $y(q) = 0$ such that $y \circ f \circ x^{-1}$ takes the form $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0)$. Since the inverse $f(M) \to M$ is also continuous, we are free to assume after possibly shrinking $\mathcal{V} \subset N$ to a smaller neighborhood of q that

$$f^{-1}(\mathcal{V} \cap f(M)) \subset \mathcal{U},$$

or in other words, $\mathcal{V} \cap f(M) = f(\mathcal{U})$. This proves that $(\mathcal{V}, y)$ is a slice chart for the subset $f(M)$. ☐

In fact, we can give an alternative definition of the notion of a submanifold as a corollary of the previous theorem.

Corollary 7.27. *A subset $L \subset M$ of a smooth manifold M is a smooth submanifold if and only if it admits a smooth structure for which the inclusion map $L \hookrightarrow M$ is a smooth embedding.* ☐

We now see the relations between submersions and submanifolds. We make the following

Definition 7.28. Let $f : M \to N$ be a smooth map. A point $p \in M$ is called a **regular point** of f if f is a submersion at p, and a **critical point** otherwise. A point $q \in N$ is a **critical value** of f if $q = f(p)$ for some critical point p, and q is otherwise called a **regular value** of f.

Theorem 7.29. *For any smooth map $f : M^m \to N^n$ with regular value $q \in N$, $L := f^{-1}(q) \subset M$ is a smooth submanifold with $\dim L = \dim M - \dim N$, and the tangent space at any point $p \in L$ is $T_pL = \ker df_p \subset T_pM$.*

Proof. We would like to find slice charts for L. For each $p \in L = f^{-1}(q)$, f is by assumption a submersion at p, so Theorem 7.11 provides charts x near p and y near q such that $x(p)$ and $y(q)$ are both the origin in their respective Euclidean spaces and $y \circ f \circ x^{-1}$ becomes the map $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^n)$. The zero-set of this map is a neighborhood of p in $f^{-1}(q)$ as seen in the x-coordinates, thus x is a slice chart.

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[2] Recall that if both L and M are manifolds of class C^k but $k < \infty$, then it does not make sense to say that the inclusion $L \hookrightarrow M$ is smooth, even though it looks smooth in the particular local coordinates we chose. The point is that one could also choose different coordinates in which it would still appear to be a map of class C^k, but not necessarily C^∞.

We now look at the tangent space and prove that $T_pL = \ker df_p$. Notice first that for any path γ in L through p, $f \circ \gamma$ is a constant path at $q \in N$, and hence $df_p([\gamma]) = 0 \in T_qN$, proving $T_pL \subset \ker df_p$. Now q is a regular value and hence $df_p : T_pM \to T_qN$ is surjective. This implies

$$\dim T_pL = \dim L = \dim M - \dim N = \dim T_pM - \dim T_qN = \dim \ker T_pf,$$

by the rank-nullity theorem. □

Definition 7.30. Submanifolds of the form $f^{-1}(q) \subset M$ for regular values $q \in N$ are called **regular level sets** of f.

Thus, a submersion $f : M \to N$ is distinguished by the property that *all* of its level sets are regular, and are thus smooth submanifolds.

In fact, the proof of Theorem 7.29 also proves the following result on maps of constant rank.

Theorem 7.31 (Constant-Rank Level set theorem)**.** *Let $f : M^m \to N^n$ be a smooth map between smooth manifolds such that f is of constant rank r. Then the level sets of f, i.e., $f^{-1}(q) \subset M$, $q \in N$ are smooth submanifolds of M of dimension $m - r$.*

We now have a *very* easy way of proving that simple examples like the unit spheres $S^n \subset \mathbb{R}^{n+1}$ really are smooth submanifolds.

Example 7.32. Define $f : \mathbb{R}^{n+1} \to \mathbb{R}$ in terms of the standard Euclidean inner product by $f(x) = |x|^2 = \langle x, x \rangle$. This is a smooth map, with differential at any point $x \in \mathbb{R}^{n+1}$ given by $d_xf(v) = 2\langle x, v \rangle$, so it is a submersion everywhere except at the origin. This makes $S^n = f^{-1}(1)$ into a smooth submanifold of dimension $(n+1) - 1 = n$, so in particular, S^n inherits a natural smooth structure for which the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth embedding. The kernel of d_xf at a point $x \in S^n$ is the orthogonal complement of x, hence

$$T_xS^n = x^\perp \subset \mathbb{R}^{n+1}.$$

which is something you prove explicitly in Problem 3 in Problem Set 3.

Example 7.33. The smooth map $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto xy$ has only one critical point, at $(x, y) = (0, 0)$, thus $f^{-1}(t)$ is a smooth submanifold (a hyperbola) for every $t \neq 0$, and so is $f^{-1}(0) \setminus \{(0, 0)\}$, but $f^{-1}(0)$ fails to be a submanifold at the origin.