We already saw the definition of a smooth local diffeomorphism. We also saw that if $f : M \to N$ is a local diffeomorphism then the linear map $df_p : T_pM \to T_{f(p)}N$ is an isomorphism. The converse of this statement is also true and is known as the **inverse function theorem.**

Theorem 7.6 (Inverse Function Theorem for Manifolds)**.** *Suppose M and N are smooth manifolds, and $f\colon M \to N$ is a smooth map. If $p \in M$ is a point such that df_p is an isomorphism, then there are neighbourhoods U_0 of p and V_0 of $f(p)$ such that $f|_{U_0}\colon U_0 \to V_0$ is a diffeomorphism.*

Proof. The fact that df_p is bijective implies that M and N have the same dimension, say n. Choose smooth charts (U, φ) centered at p and (V, ψ) centered at $f(p)$, with $f(U) \subseteq V$. Then $\widehat{f} = \psi \circ f \circ \varphi^{-1}$ is a smooth map from the open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$ into $\widehat{V} = \psi(V) \subseteq \mathbb{R}^n$, with $\widehat{f}(p) = 0$. Because φ and ψ are diffeomorphisms, the differential $d\widehat{f}_0 = d\psi_{f(p)} \circ df_p \circ d(\varphi^{-1})_0$ is nonsingular. The inverse function theorem for Euclidean spaces implies that there are connected open subsets $\widehat{U}_0 \subseteq \widehat{U}$ and $\widehat{V}_0 \subseteq \widehat{V}$ containing 0 such that $\widehat{f}$ restricts to a diffeomorphism from $\widehat{U}_0$ to $\widehat{V}_0$. Then $U_0 = \varphi^{-1}(\widehat{U}_0)$ and $V_0 = \psi^{-1}(\widehat{V}_0)$ are connected neighborhoods of p and $f(p)$, respectively, and it follows by composition that $f|_{U_0}$ is a diffeomorphism from U_0 to V_0. □

The next result shows the relationship between immersions, submersions and local diffeomorphisms.

Proposition 7.7. *Suppose M and N are smooth manifolds and let $f\colon M \to N$ be a map.*

1. *f is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.*
2. *If $\dim M = \dim N$ and f is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.*

Proof. Suppose first that f is a local diffeomorphism. Given $p \in M$, there is a neighbourhood U of p such that f is a diffeomorphism from U to $f(U)$ and $df_p\colon T_pM \to T_{f(p)}N$ is an isomorphism. Thus $\operatorname{rank} f = \dim M = \dim N$, so f is both a smooth immersion and a smooth submersion. Conversely, if f is both a smooth immersion and a smooth submersion, then df_p is an isomorphism at each $p \in M$, and the inverse function theorem Theorem refthm:IFT shows that p has a neighborhood on which f restricts to a diffeomorphism onto its image. This proves 1.

To prove 2., note that if M and N have the same dimension, then either injectivity or surjectivity of df_p implies bijectivity, so f is a smooth submersion if and only if it is a smooth immersion, and thus 2 follows from 1. □

7.2 The Rank Theorem

We see example of a submersion and immersion.

Example 7.8. Canonical Projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$ (where $m > n$)

$$\pi(x^1, x^2, \ldots, x^m) = (x^1, x^2, \ldots, x^n)$$

is a submersion. The differential at any point p is the linear map:

$$d\pi_p : \mathbb{R}^m \to \mathbb{R}^n$$

represented by the $n \times m$ matrix

$$d\pi_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

This matrix has rank n everywhere $\implies$ $d\pi_p$ is surjective $\implies$ π is a submersion. Also notice that each fiber $\pi^{-1}(c) \cong \mathbb{R}^{m-n}$ is an affine subspace.

Example 7.9. The Canonical Inclusion $\iota : \mathbb{R}^m \to \mathbb{R}^n$ (where $m < n$)

$$\iota(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0)$$

is an immersion. The differential at any point p is:

$$d\iota_p = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in M_{n\times m}(\mathbb{R})$$

This is an $n \times m$ matrix of rank m everywhere $\implies d\iota_p$ is injective.

The most important fact about constant-rank maps is the following consequence of the inverse function theorem, which says that a constant-rank smooth map can be placed locally into a particularly simple canonical form by a change of coordinates and so in a way the examples provided above of submersions and immersions are the model examples.

We first recall the following canonical form result for surjective linear maps.

Theorem 7.10 (Canonical Form for a Linear Map)**.** *Suppose V and W are finite-dimensional vector spaces, and $T : V \to W$ is a linear map of rank r. Then there are bases for V and W with respect to which T has the following matrix representation*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix.

Theorem 7.11 (Rank Theorem). *Suppose M and N are smooth manifolds of dimensions m and n, respectively, and $F\colon M \to N$* ***is a smooth map with constant rank*** *r. For each $p \in M$ there exist smooth charts $(U, \varphi) \ni p$ for M and $(V, \psi) \ni F(p)$ for N such that $F(U) \subseteq V$, in which F* ***has a coordinate representation of the form***

$$\widehat{F}\big(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m\big) = \big(x^1, \ldots, x^r, 0, \ldots, 0\big). \tag{7.1}$$

If F is a smooth submersion then the representation becomes

$$\widehat{F}\big(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m\big) = \big(x^1, \ldots, x^n\big), \tag{7.2}$$

and if F is a smooth immersion, it is

$$\widehat{F}\big(x^1, \ldots, x^m\big) = \big(x^1, \ldots, x^m, 0, \ldots, 0\big). \tag{7.3}$$

Proof. Because the theorem is local, after choosing smooth coordinates we can replace M and N by open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. Since $DF(p)$ has rank r hence its matrix representation has some $r \times r$ submatrix with nonzero determinant. By reordering the coordinates, w.l.o.g., we assume that it is the upper left submatrix, $(\partial F^i/\partial x^j)$ for $i, j = 1, \ldots, r$. We write the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$ as $(x, y) = (x^1, \ldots, x^r, y^1, \ldots, y^{m-r})$ in $\mathbb{R}^m$ and $(v, w) = (v^1, \ldots, v^r, w^1, \ldots, w^{n-r})$ in $\mathbb{R}^n$. By initial translations of the coordinates, we may assume without loss of generality that $p = (0, 0)$ and $F(p) = (0, 0)$.

If we write $F(x, y) = \big(Q(x, y), R(x, y)\big)$ for some smooth maps $Q\colon U \to \mathbb{R}^r$ and $R\colon U \to \mathbb{R}^{n-r}$, then our hypothesis is that $(\partial Q^i/\partial x^j)$ is nonsingular at $(0, 0)$.

Define $\varphi\colon U \to \mathbb{R}^m$ by $\varphi(x, y) = \big(Q(x, y), y\big)$. Its total derivative at $(0, 0)$ is

$$D\varphi(0,0) = \begin{pmatrix} \dfrac{\partial Q^i}{\partial x^j}(0,0) & \dfrac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & \delta^i_j \end{pmatrix},$$

where δ^i_j is the **Kronecker delta**, which is defined by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The matrix $D\varphi(0, 0)$ is nonsingular because one block $\frac{\partial Q^i}{\partial x^j}(0, 0)$ is non-singular at $(0, 0)$ and the second diagonal block is just the identity matrix. Therefore, by the inverse function theorem Theorem 7.6, there are connected neighbourhoods U_0 of $(0,0)$ and $\widetilde{U}_0$ of $\varphi(0,0) = (0,0)$ such that $\varphi\colon U_0 \to \widetilde{U}_0$ is a diffeomorphism. By shrinking U_0 and $\widetilde{U}_0$ if necessary, we may assume that $\widetilde{U}_0$ is an open cube.

Writing the inverse map as $\varphi^{-1}(x,y) = \big(A(x,y), B(x,y)\big)$ for some smooth functions $A\colon \widetilde{U}_0 \to \mathbb{R}^r$ and $B\colon \widetilde{U}_0 \to \mathbb{R}^{m-r}$, we compute

$$(x,y) = \varphi\big(A(x,y), B(x,y)\big) = \big(Q\big(A(x,y), B(x,y)\big), B(x,y)\big). \tag{7.4}$$

Comparing y components shows that $B(x,y) = y$, and therefore φ^{-1} has the form

$$\varphi^{-1}(x,y) = \big(A(x,y), y\big).$$

On the other hand, $\varphi \circ \varphi^{-1} = \mathrm{Id}$ implies $Q\big(A(x,y), y\big) = x$, and therefore $F \circ \varphi^{-1}$ has the form

$$F \circ \varphi^{-1}(x,y) = \big(x, \widetilde{R}(x,y)\big),$$

where $\widetilde{R}\colon \widetilde{U}_0 \to \mathbb{R}^{n-r}$ is defined by $\widetilde{R}(x,y) = R\big(A(x,y), y\big)$. The Jacobian matrix of this composite map at an arbitrary point $(x,y) \in \widetilde{U}_0$ is

$$D(F \circ \varphi^{-1})(x,y) = \begin{pmatrix} \delta^i_j & 0 \\ \dfrac{\partial \widetilde{R}^i}{\partial x^j}(x,y) & \dfrac{\partial \widetilde{R}^i}{\partial y^j}(x,y) \end{pmatrix}.$$

Since composing with a diffeomorphism does not change the rank of a map, this matrix has rank r everywhere in $\widetilde{U}_0$ because F was assumed to have rank r and φ^{-1} is a diffeomorphism. The first r columns are obviously linearly independent, so the rank can be r only if the derivatives $\partial \widetilde{R}^i / \partial y^j$ vanish identically on $\widetilde{U}_0$, which implies that $\widetilde{R}$ is actually independent of $(y^1, \dots, y^{m-r})$. Thus, if we let $S(x) = \widetilde{R}(x, 0)$, then we have

$$F \circ \varphi^{-1}(x,y) = \big(x, S(x)\big). \tag{7.5}$$

We complete the proof by defining an appropriate smooth chart in some neighborhood of $(0,0) \in V$. Let $V_0 \subseteq V$ be the open subset defined by $V_0 = \{(v,w) \in V : (v,0) \in \widetilde{U}_0\}$. Then V_0 is a neighborhood of $(0,0)$. Because $\widetilde{U}_0$ is a cube and $F \circ \varphi^{-1}$ has the form (7.5), it follows that $F \circ \varphi^{-1}(\widetilde{U}_0) \subseteq V_0$, and therefore $F(U_0) \subseteq V_0$. Define $\psi\colon V_0 \to \mathbb{R}^n$ by $\psi(v,w) = \big(v, w - S(v)\big)$. This is a diffeomorphism onto its image, because its inverse is given explicitly by $\psi^{-1}(s,t) = \big(s, t + S(s)\big)$; thus (V_0, ψ) is a smooth chart. It follows from (7.5) that

$$\psi \circ F \circ \varphi^{-1}(x,y) = \psi\big(x, S(x)\big) = \big(x, S(x) - S(x)\big) = (x, 0),$$

which was to be proved. □

The next corollary can be viewed as a more invariant statement of the rank theorem. It says that constant-rank maps are precisely the ones whose local behaviour is the same as that of their differentials.

Corollary 7.12. *Let M and N be smooth manifolds, let $F : M \to N$ be a smooth map, and suppose M is connected. Then the following are equivalent:*

(a) For each $p \in M$ there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.

(b) F has constant rank.

Proof. First suppose F has a linear coordinate representation in a neighbourhood of each point. Since every linear map has constant rank, it follows that the rank of F is constant in a neighbourhood of each point, and thus by connectedness it is constant on all of M. Conversely, if F has constant rank, the rank theorem shows that it has the linear coordinate representation (7.1) in a neighbourhood of each point. □

We now use the rank theorem, which is purely a local statement to say something about the global nature of smooth maps based on their rank. Before we do that, we recall the following topological notions.

Definition 7.13. Let X be a topological space.

(a) A subset $A \subseteq X$ is called **nowhere dense** if its closure has empty interior:

$$\operatorname{int}(\overline{A}) = \emptyset.$$

(b) A subset $A \subseteq X$ is called **meagre** (or **of the first category**) if it can be written as a countable union of nowhere dense sets:

$$A = \bigcup_{n=1}^{\infty} A_n, \qquad \operatorname{int}(\overline{A_n}) = \emptyset \quad \forall\, n \in \mathbb{N}.$$

(c) A subset $A \subseteq X$ is called **comeagre** (or **residual**, or **of the second category**) if its complement $X \setminus A$ is meagre.

Theorem 7.14 (Baire Category Theorem)**.** *The following two classes of topological spaces are* ***Baire spaces****, meaning that every countable intersection of open dense subsets is dense:*

(i) Every ***complete metric space*** (X, d) *is a Baire space.*

(ii) Every ***locally compact Hausdorff space*** X *is a Baire space.*

Another may wo reformulate this is the following. Let X be either a complete metric space or a locally compact Hausdorff space. Then the following equivalent statements hold:

(a) ***(Open dense sets)*** *If $\{U_n\}_{n=1}^{\infty}$ is a countable collection of open dense subsets of X, then*

$$\bigcap_{n=1}^{\infty} U_n \neq \emptyset.$$

In fact, $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

(b) ***(Nowhere dense sets)*** *If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of nowhere dense subsets of X, then*

$$\operatorname{int}\left(\bigcup_{n=1}^{\infty} A_n\right) = \emptyset.$$

That is, X cannot be written as a countable union of nowhere dense sets.

(c) ***(Meagre sets)*** *No nonempty open subset of X is meagre. Equivalently, X itself is not meagre:*

$$X \neq \bigcup_{n=1}^{\infty} A_n \quad \text{whenever each } A_n \text{ is nowhere dense in } X.$$

(d) ***(Residual sets)*** *Every comeagre (residual) subset of X is dense in X.*

We now state and prove the Global Rank theorem.

Theorem 7.15 (Global Rank Theorem)**.** *Let M and N be smooth manifolds, and suppose $F : M \to N$ is a smooth map of constant rank.*

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

Proof. Let $m = \dim M$, $n = \dim N$, and suppose F has constant rank r. To prove (a), assume that F is not a smooth submersion, which means that $r < n$. By the rank theorem, Theorem 7.11, for each $p \in M$ there are smooth charts

(U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$ and the coordinate representation of F is given by (7.1).

Shrinking U if necessary, we may assume that it is a regular coordinate ball and $F(\bar{U}) \subseteq V$, where $\bar{U}$ is the closed coordinate ball and is the closure of the open set U. This implies that $F(\bar{U})$ is a compact subset of the set $\{y \in V : y^{r+1} = \cdots = y^n = 0\}$, so it is closed in N and contains no open subset of N; hence it is nowhere dense in N. Since every open cover of a manifold has a countable subcover, we can choose countably many such charts $\{(U_i, \varphi_i)\}$ covering M, with corresponding charts $\{(V_i, \psi_i)\}$ covering $F(M)$. Because $F(M)$ is equal to the countable union of the nowhere dense sets $F(\bar{U}_i)$, it follows from the Baire category theorem, Theorem 7.14, that $F(M)$ has empty interior in N, which means F cannot be surjective.

To prove (b), assume that F is not a smooth immersion, so that $r < m$. By the rank theorem, for each $p \in M$ we can choose charts on neighbourhoods of p and $F(p)$ in which F has the coordinate representation (7.1). It follows that $F(0, \ldots, 0, \varepsilon) = F(0, \ldots, 0, 0)$ for any sufficiently small ε, so F is not injective.

Finally, (c) follows from (a) and (b), because a bijective smooth map of constant rank is a smooth submersion by part (a) and a smooth immersion by part (b); so Proposition 7.7 implies that F is a local diffeomorphism, and because it is bijective, it is a diffeomorphism. □

7.3 Embeddings

One special kind of immersion is particularly important.

Definition 7.16. If M and N are smooth manifolds, a ***smooth embedding of*** M ***into*** N is a smooth immersion $F : M \to N$ that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.

Example 7.17 (Smooth Embeddings).

1. If M is a smooth manifold with and $U \subseteq M$ is an open submanifold, the inclusion map $U \hookrightarrow M$ is a smooth embedding.

2. If $M_1, \ldots, M_k$ are smooth manifolds and $p_i \in M_i$ are arbitrarily chosen points, each of the maps $\iota_j : M_j \to M_1 \times \cdots \times M_k$ given by
$$\iota_j(q) = (p_1, \ldots, p_{j-1}, q, p_{j+1}, \ldots, p_k)$$
is a smooth embedding. In particular, the inclusion map $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+k}$ given by sending $(x^1, \ldots, x^n)$ to $(x^1, \ldots, x^n, 0, \ldots, 0)$ is a smooth embedding.