In fact, this also allows us to prove the following result which tells that the tangent vectors as equivalence classes is the same as derivations.

Lemma 5.5. *Let M be a differentiable manifold and let $p \in M$. Then the map*

$$\partial : T_pM \to \mathrm{Der}(C_p^\infty), \qquad \gamma'(0) \mapsto \partial_{\gamma'(0)},$$

is an **isomorphism.** *In particular, every derivation is a directional derivative and for every chart $\phi : U \to V$ with $p \in U$*

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

is a basis of $\mathrm{Der}(C_p^\infty)$.

Proof. We have already proved that ∂ is indeed a linear map between vector spaces and that given a chart, we get the derivations $\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$. We now show that

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

form a basis of $\mathrm{Der}(C_p^\infty)$. Namely, then we know that the linear map ∂ maps the basis

$$(d\phi|_p)^{-1}(e_1), \dots, (d\phi|_p)^{-1}(e_n)$$

of T_pM onto the basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$$

of $\mathrm{Der}(C_p^\infty)$ and is hence an isomorphism.

a) **Linear Independence:** Let

$$\sum_{i=1}^n \alpha^i \left.\frac{\partial}{\partial x^i}\right|_p = 0.$$

We have to show:

$$\alpha^1 = \cdots = \alpha^n = 0.$$

Choose

$$f = x^j.$$

Then

$$0 = \sum_{i=1}^n \alpha^i \left.\frac{\partial x^j}{\partial x^i}\right|_p = \alpha^j \qquad \text{for } j = 1, \dots, n.$$

b) **Spanning set:** Let $\delta \in \mathrm{Der}(C_p^\infty)$. Set

$$\alpha^j := \delta(x^j) = \delta(\phi^j), \qquad \text{for } j = 1, \dots, n.$$

We will show that

$$\delta = \sum_{j=1}^n \alpha^j \cdot \left.\frac{\partial}{\partial x^j}\right|_p .$$

Exercise: Prove that any derivation vanish on constant functions.

Let $f \in C_p^\infty$, and assume, without loss of generality that

$$\phi(\tilde{\tilde{U}}) = B(\phi(p), r).$$

More precisely, since $f \in C^\infty(\tilde{U})$ with $p \in \tilde{U}$ open, so we choose a neighborhood $\tilde{\tilde{U}}$ of p with

$$\tilde{\tilde{U}} \subset \tilde{U} \cap U$$

and assume that its image is an open ball. If M were $\mathbb{R}^n$ then to produce a function we would have used Taylor's theorem. In fact,

Claim 5.6. *For any function $h \in C^\infty(B(q,r))$, there exist $g_1, \dots, g_n \in C^\infty(B(q,r))$ with*

(i)

$$h(x) = h(q) + \sum_{i=1}^{n} (x^i - q^i) g_i(x)$$

and

(ii)

$$\frac{\partial h}{\partial x^i}(q) = g_i(q).$$

Proof of the Claim: For $x \in B(q,r)$ set

$$w_x : [0,1] \to \mathbb{R}, \qquad w_x(t) := h(tx + (1-t)q).$$

Then,

$$\begin{aligned} h(x) - h(q) &= w_x(1) - w_x(0) \\ &= \int_0^1 \dot{w}_x(t)\, dt \qquad \text{(Fundamental theorem of Calculus)} \\ &= \int_0^1 \sum_{i=1}^{n} \frac{\partial h}{\partial x^i}\bigg|_{tx+(1-t)q} \cdot (x^i - q^i)\, dt \\ &= \sum_{i=1}^{n} (x^i - q^i) \underbrace{\int_0^1 \frac{\partial h}{\partial x^i}\bigg|_{tx+(1-t)q} dt}_{=:g_i(x)}. \end{aligned}$$

We now work backwards and define g_i as the function in the last line of the equation, thus proving (i) . Once we have (i), (ii) follows via differentiation. ☐

We use the claim proved above with $h = f \circ \phi^{-1}$. Thus, there exist $g_1, \dots, g_n \in C^\infty(B(x(p),r))$ such that

$$(f \circ \phi^{-1})(x) = (f \circ \phi^{-1})(\phi(p)) + \sum_{i=1}^{n} \big(\phi^i - \phi^i(p)\big) \cdot g_i(x)$$

and

$$\frac{\partial (f \circ \phi^{-1})}{\partial x^i}(\phi(p)) = g_i(\phi(p)).$$

It follows that

$$\begin{aligned} \delta(f) &\overset{\text{locality}}{=} \delta\big(f|_{\tilde{U}}\big) \\ &= \delta\left(f(p) + \sum_{i=1}^{n} \big(\phi^i - \phi^i(p)\big)(g_i \circ \phi) \right) \\ &= \sum_{i=1}^{n} \delta\big((\phi^i - \phi^i(p))(g_i \circ \phi)\big) \\ &\overset{\text{Leibniz rule}}{=} \sum_{i=1}^{n} \left(\delta(\phi^i - \phi^i(p)) \cdot g_i(\phi(p)) + \underbrace{(\phi^i - \phi^i(p))|_p}_{=0} \delta(g_i \circ \phi) \right) \\ &\overset{\text{linearity}}{=} \sum_{i=1}^{n} \delta(x^i)\, g_i(x(p)) \\ &= \sum_{i=1}^{n} \alpha^i \cdot \frac{\partial f}{\partial x^i}\bigg|_p . \end{aligned}$$

☐

From now on we identify T_pM with $\mathrm{Der}(C_p^\infty)$ via the isomorphism ∂. For example, we write for $\xi \in T_pM$

$$\xi = \sum_{i=1}^{n} \xi^i \left.\frac{\partial}{\partial x^i}\right|_p$$

instead of

$$\partial_\xi = \sum_{i=1}^{n} \xi^i \left.\frac{\partial}{\partial x^i}\right|_p \qquad \text{and} \qquad \xi = \sum_{i=1}^{n} \xi^i (d\phi|_p)^{-1}(e_i)$$

where

$$(\xi^1, \dots, \xi^n)^\top = d\phi|_p(\xi).$$

Change of coordinates:

How do the coefficients $\xi^1, \dots, \xi^n$ of a tangent vector change, if we replace the chart (U, ϕ) by another chart (V, ψ)?

Let $\xi \in T_pM$, let (U, ϕ) and (V, ψ) be charts, both containing p. We express ξ with respect to both charts

$$\xi = \sum_{i=1}^{n} \xi^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^{n} \eta^j \left.\frac{\partial}{\partial y^j}\right|_p .$$

Now we want to compute the coefficients ξ^i in terms of the η^j and vice versa. Using the Chain Rule, Prop 4.12, we compute

$$\begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix} = d\phi|_p(\xi) = (d\phi|_p)\left((d\psi|_p)^{-1} \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix} \right) = D(\phi \circ \psi^{-1})|_{\psi(p)} \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix} .$$

Similarly,

$$\begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix} = D(\psi \circ \phi^{-1})|_{\phi(p)} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix} .$$

Thus

$$\eta^j = \sum_{i=1}^{n} \left.\frac{\partial(\psi^j \circ \phi^{-1})}{\partial x^i}\right|_{phi(p)} \cdot \xi^i \tag{1.6}$$

Let us look at the special case

$$\xi = \left.\frac{\partial}{\partial x^i}\right|_p ,$$

that is,

$$(\xi^1, \dots, \xi^n)^\top = e_i.$$

We get

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p &= \sum_{j=1}^{n} \eta^j \left.\frac{\partial}{\partial y^j}\right|_p \\ &= \sum_{j=1}^{n} \sum_{k=1}^{n} \xi^k \left.\frac{\partial(y^j \circ x^{-1})}{\partial x^k}\right|_{x(p)} \cdot \left.\frac{\partial}{\partial y^j}\right|_p \\ &= \sum_{j=1}^{n} \left.\frac{\partial(y^j \circ x^{-1})}{\partial x^i}\right|_{x(p)} \cdot \left.\frac{\partial}{\partial y^j}\right|_p , \end{aligned}$$

hence

$$\left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^{n} \left.\frac{\partial(y^j \circ x^{-1})}{\partial x^i}\right|_{x(p)} \cdot \left.\frac{\partial}{\partial y^j}\right|_p \tag{1.7}$$

If we use the Einstein summation convention, meaning that when an index appears twice in an expression, once as an upper index and once as a lower index, then summation over this index is understood, then the above equation could be written as

$$\left.\frac{\partial}{\partial x^i}\right|_p = \left.\frac{\partial(y^j \circ x^{-1})}{\partial x^i}\right|_{x(p)} \cdot \left.\frac{\partial}{\partial y^j}\right|_p$$

or even shorter as

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial}{\partial y^j}.$$

5.1 Differential of a map in coordinates

Recall that we defined the differential of a map f in terms of using the notion of tangent vectors via curves as well as using derivations. We would like to do some explicit computations and would like to see how the term df for some $f : M \to N$ looks like in coordinates. Let's try to understand this in the spacial case when $M = \mathbb{R}^n$ and $N = \mathbb{R}^m$. Let $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$ be a smooth map. For any $p \in U$, we expect df to be a $m \times n$ matrix. We determine the matrix explicitly, in terms of the standard coordinate bases. Using $(x^1, \ldots, x^n)$ to denote the coordinates in the domain and $(y^1, \ldots, y^m)$ to denote those in the codomain, we use the chain rule to compute the action of df_p on a typical basis vector as follows:

$$\begin{aligned} df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) h &= \left.\frac{\partial}{\partial x^i}\right|_p (h \circ f) \\ &= \frac{\partial h}{\partial y^j}(f(p))\,\frac{\partial f^j}{\partial x^i}(p) \\ &= \left(\frac{\partial f^j}{\partial x^i}(p)\,\left.\frac{\partial}{\partial y^j}\right|_{f(p)}\right) h. \end{aligned}$$

Thus,

$$df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial f^j}{\partial x^i}(p)\,\left.\frac{\partial}{\partial y^j}\right|_{f(p)}. \tag{5.1}$$

This says that the matrix of the linear map df_p in terms of the coordinate bases is

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1}(p) & \cdots & \frac{\partial f^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x^1}(p) & \cdots & \frac{\partial f^m}{\partial x^n}(p) \end{pmatrix},$$

which is precisely the Jacobian matrix of f at p. Therefore, in this case,

$$df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$$

corresponds to the total derivative

$$Df(p) : \mathbb{R}^n \to \mathbb{R}^m.$$

Now consider the more general case of a smooth map $F : M \to N$ between smooth manifolds. Choosing smooth coordinate charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation

$$\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V).$$

Let $\hat{p} = \varphi(p)$ denote the coordinate representation of p. By the computation above, $d\hat{F}_{\hat{p}}$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\hat{F}$ at $\hat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$, we compute

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = dF_p\left(d(\varphi^{-1})_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) = d(\psi^{-1})_{\hat{F}(\hat{p})}\left(d\hat{F}_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right).$$

latex

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= d(\psi^{-1})_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\,\left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\hat{p})}\right) \\ &= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\,\left.\frac{\partial}{\partial y^j}\right|_{F(p)}. \end{aligned} \tag{3.10}$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representative of) F. Sometimes, the differential of a map is also called the **pushforward of F** and is denoted as F_*.

6. Vector fields

Next we want to introduce vector fields. Vector fields are maps which associate to each point of a manifold a tangent vector in the corresponding tangent space. We have the following:

Definition 6.1. A map $\xi : M \to TM$ is called a **vector field** on M, if for every $p \in M$ we have

$$\pi(\xi(p)) = p. \tag{6.1}$$

We can use the coordinate representation of a chart to write the local coordinate description of any vector field. Let $\phi : U \to V$ be a chart of M. A vector field ξ on U is characterized by the functions which are coefficients of the tangent vector $\xi(p),\ p \in M$. More precisely, we have

$$\xi^1, \dots, \xi^n : V \to \mathbb{R}$$

for which

$$\xi(p) = \sum_{i=1}^{n} \xi^i(\phi(p)) \left. \frac{\partial}{\partial x^i} \right|_p .$$

For the chart ϕ of M we consider the corresponding chart Φ_ϕ on TM. Using the notation as before, we see that ξ corresponds to the map

$$v \mapsto (v, \xi^1(v), \dots, \xi^n(v)), \ \ v \in V = \phi(U) \subset \mathbb{R}^n.$$

Thus ξ is C^k on U if and only if the coefficient functions $\xi^1, \dots, \xi^n$ are C^k on V.

7. Submanifolds

As with any mathematical structure, once we have seen and understood the theory of manifolds, it makes sense to ask whether subsets of manifolds also admit a smooth structure. We will see that the notion of *immersions, submersions and embeddings* are the key properties which allow us to study special subsets of a manifold which are manifolds in their own right.

7.1 Submersions, immersions and embeddings

Definition 7.1. Let M and N are smooth manifolds and let $F\colon M \to N$ be a smooth point. For $p \in M$, we define the **rank of F at p** to be the rank of the linear map

$$dF_p\colon T_pM \to T_{F(p)}N.$$

In other words, it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of $\operatorname{Im} dF_p \subseteq T_{F(p)}N$.
If F has the same rank r at every point, we say that it has **constant rank**, and write $\operatorname{rank} F = r$.

We know by the rank-nullity theorem that rank of $F \leq \min\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, we say that F **has full rank at** p, and if F has full rank everywhere, we say F **has full rank**.

Definition 7.2. A smooth map $F\colon M \to N$ is called a **smooth submersion** if its differential is surjective at each point (or equivalently, if $\operatorname{rank} F = \dim N$).

It is called a **smooth immersion** if its differential is injective at each point (equivalently, $\operatorname{rank} F = \dim M$).

Proposition 7.3. *Suppose $F\colon M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. If dF_p is injective, then p has a neighborhood U such that $F|_U$ is an immersion.*

Proof. Choose any smooth coordinates for M near p and for N near $F(p)$, either hypothesis means that Jacobian matrix of F in coordinates has full rank at p. The result now follows from the fact that the set of $m \times n$ matrices of full rank is an open subset of $\mathrm{M}(m \times n, \mathbb{R})$[1], so by continuity, the Jacobian of F has full rank in some neighbourhood of p. ☐

Example 7.4 (Submersions and Immersions).

(a) Suppose $M_1, \dots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i\colon M_1 \times \cdots \times M_k \to M_i$ is a smooth submersion. In particular, the projection $\pi\colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a smooth submersion.

(b) If $\gamma\colon J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

(c) Exercise: If M is a smooth manifold and its tangent bundle is denoted by TM prove that the projection $\pi\colon TM \to M$ is a smooth submersion.

(d) The smooth map $X\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u,v) = \big((2+\cos 2\pi u)\cos 2\pi v,\ (2+\cos 2\pi u)\sin 2\pi v,\ \sin 2\pi u\big)$$

is a smooth immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the doughnut-shaped surface obtained by revolving the circle $(y-2)^2 + z^2 = 1$ in the (y,z)-plane about the z-axis.

Exercise 7.5. Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

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[1] Let f be the function taking $A \in M(m \times n)$ to the $f(A) = \Sigma_B |\det(B)|$ where the sum is running over all $m \times m$ (assuming $m < n$) sub-matrices of A. Clearly, f is continuous so $f^{-1}(\mathbb{R} \setminus \{0\})$ is an open set is $M(m \times n)$